Chromatic Lecture 12: Bousfield localizations

References:
- Chapter 6 of the TMF book.
- Lecture 20 of Lurie's notes.
- Survey by Tyler Lawson.

§1. E-acyclic & local spectra.

§2. Construction & Properties of Bousfield Localizations.

§3. Examples.

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1. E-acyclic & local spectra.

Motivation: In Alg Top, we want to classify spaces up to htpy. or weak equivalences.

Recall: $f: X \to Y$ is a wkEq if $f_*: \pi_n(X) \to \pi_n(Y)$ isoms $\forall n$.

WkEqs are the equivalences detected by htpy gps.

Obs: stable htpy gp is a generalized homology thy, represented by $\mathbb{S}$.

Question: What if we replace $\mathbb{S}$ by some generalized homology theories.

This leads to

Defn: Let E be a generalized homology theory.

(1) A spectrum X is called E-acyclic if $E_*(X)=0 \Leftrightarrow E\wedge X \simeq *$

(2) $X \in Sp$ is called E-local if $\forall T$ E-acyclic, $[T, X]=0$

(3). A map of spectra $f: X \to Y$ is called E-equivalence if $f_*: E_*(X) \xrightarrow{\sim} E_*(Y)$ $\Leftrightarrow 1\wedge f: E\wedge X \xrightarrow{\sim} E\wedge Y$.

Ex: (1). $E = H\mathbb{Z}$, then X is E-acyclic if $E_* X = 0$ e.g. $X = K(n)$.

- Any Eilenberg-MacLane spectrum HA is E-local.

Pf: If T is acyclic, then

$$[T, HA] = H^0(T; A) \underset{\substack{\uparrow \\ \text{By Univ coeff thm.}}}{=} 0$$

(2). $E = H\mathbb{Q}$, then $\mathbb{RP}^2$ is E-acyclic.

b/c: $\widetilde{H}_*(\mathbb{RP}^2; \mathbb{Q}) = 0$.

Any Moore spectrum $M(A)$ A is finite gp is E-acyclic.

Rmk: Analogy w/ commutative algebra.

R = comm alg. $M \in R$-mod $f \in R$.

- M is f-torsion if $M \xrightarrow{f} M$ is zero.

$\Longleftrightarrow$ $M \otimes_R R[\frac{1}{f}] = 0$. M is "$R[\frac{1}{f}]$-acyclic"

- If $\forall$ f-torsion M', $\mathrm{Hom}_R(M', M) = 0$.

$\Longleftarrow$ $M \xrightarrow{f} M$ isom. M is f-local.

Lem: A spectrum X is E-local iff $\forall$ E-equiv $f: Y_1 \to Y_2$,

$$f^* : \mathrm{Map}(Y_2, X) \longrightarrow \mathrm{Map}(Y_1, X).$$

is an equivalence.

pf: $f: Y_1 \to Y_2$ is E-equiv.

$\Rightarrow$ $\mathrm{cofib}(f)$ is E-acyclic.

$\mathrm{Map}(Y_1 \to Y_2 \to \mathrm{cofib}(f), X)$.

$$\mathrm{Map}(\mathrm{cofib}(f), X) \longrightarrow \mathrm{Map}(Y_2, X) \longrightarrow \mathrm{Map}(Y_1, X)$$

If X is E-local, then is $*$ $\Rightarrow$ f^* is an equiv.

On the other hand. any E-acyclic can be realized as the cofiber of E-equiv's.
This implies "$\Leftarrow$".

§2. Bousfield localization

Slogan: E be a homology theory, $X \in Sp$.

then $X \to L_E X$ is the best "approximation" of X by E-local spectrum.

Defn: $L_E : Sp \to Sp$ idempotent functor

together w/ $\eta : id \Rightarrow L_E$ s.t.

$\forall X \in Sp$. $\eta : X \to L_E X$ is an E-equiv.

& $L_E X$ is E-local.

Thm (Bousfield) (L_E, η) exists.

Idea of Construction:

Sp has a model cat structure.

If we replace wkEq by E-equiv.
keep cofibrations,

then $\quad \text{Fib} = \text{Trivial cofib}^{\boxslash}$

$\quad = (\text{cofib} \cap E\text{-equiv})^{\boxslash}$

In the E-model structure

$X \to *$ is no longer a fibration.

$\exists$ factorization: $X \xrightarrow[\text{E-equiv.}]{} L_E X \xrightarrow[\text{E-fib}]{} *$

i.e. $\forall f: Y_1 \to Y_2$ an E-equiv & cofib.

$$\begin{tikzcd} Y_1 \arrow[r] \arrow[d] & L_E X \arrow[d] \\ Y_2 \arrow[r] \arrow[ur, dashed, "\exists"] & * \end{tikzcd}$$

$\rightsquigarrow$ This is precisely is lemma i.e. $L_E X$ is E-loc iff such diagram admits a lift.

Prop: Universal properties of $X \to L_E X$ is

- initial among maps from X to an E-loc target.
- terminal among E-equivs from X.

Pf: suppose $f: X \to Y$ Y E-loc.

$$\begin{tikzcd}
\mathrm{Fib} \arrow[r] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \arrow[rrd, bend right, "\boxed{0}"'] & X \arrow[r] \arrow[rd] & L_E X \arrow[d, dashed, "\exists"] \\
E\text{-acyclic.} & & Y
\end{tikzcd}$$

Prop. If E is a weak ring spectrum

i.e. E, $\mu : E \wedge E \to E$

$\varepsilon : S^0 \to E$

s.t.

$$\begin{tikzcd}
E \arrow[r, "\varepsilon \wedge 1"] \arrow[rr, bend right, "id"'] & E \wedge E \arrow[r, "\mu"] & E
\end{tikzcd}$$

then any "E-mod" M is E-local.

$(M, \alpha : E \wedge M \to M$ that is unital i.e.

$$\begin{tikzcd}
M \arrow[r, "\varepsilon \wedge 1"] \arrow[rr, bend right, "id"'] & E \wedge M \arrow[r, "\alpha"] & M
\end{tikzcd}$$

Prop: Homotopy retracts & limits of E-local spectra ~~are~~ are E-local.

Pf: X E-local.
$$\begin{tikzcd}
Y \arrow[r, "i"] \arrow[rr, bend right, "id_Y"'] & X \arrow[r, "r"] & Y
\end{tikzcd}$$

For any E-acyclic T.

$$\begin{tikzcd} \mathrm{Map}(T, Y) \arrow[r, "r^*"] \arrow[rr, bend right, "\mathrm{id}"'] & \overset{\simeq 0 \text{ b/c } X \text{ is } E\text{-local}}{\mathrm{Map}(T, X)} \arrow[r, "i^*"] & \mathrm{Map}(T, Y) \end{tikzcd}$$

$\Rightarrow$ Y is E-local.

Rmk: L_E does not preserve holims!

Defn: $Sp_E \subseteq Sp$ be the full subcat. of E-local spectra.

Prop. $E \in Sp$. TFAE:

(1). $Sp_E \subseteq Sp$ is closed under hocolim.

(2). $L_E : Sp \to Sp$ preserves hocolim.

(3). $L_E X \simeq (L_E S^0) \wedge X$

If the above is satisfied, we call L_E "smashing"

Pf: $(1) \rightleftarrows (2) \Leftrightarrow (3)$.

$(2) \Leftrightarrow (3)$ Fact: Any colim-presering

$F: Sp \to Sp$ is of the form $F(X) \simeq X \wedge T$.

2) L_E preserves colim: $L_E X \simeq X \wedge T$.

$T = L_E S^0$.

(1) $\Leftrightarrow$ (2) exercise.

§3. Examples.

Thm (Universal coeff thm for π_*).

$X \in Sp$, $A \in Ab$, $M(A)$ = Moore spectrum of A.

$\exists$ natural SES. that does not always split.

$$\begin{tikzcd} 0 \arrow[r] & \pi_n(X) \otimes_{\mathbb{Z}} A \arrow[r] & \pi_n(X \wedge M(A)) \arrow[r] \arrow[d, phantom, "\parallel"] & \mathrm{Tor}_1^{\mathbb{Z}}(A, \pi_{n-1}(X)) \arrow[d] \\ & & \pi_n(X; A). & 0 \end{tikzcd}$$

Defn: Let X be a spectrum. $f \in \pi_k \mathrm{Map}(X, X)$.

(1) $X[\frac{1}{f}] = \mathrm{hocolim}(X \xrightarrow{f} \Sigma^{-k} X \xrightarrow{\Sigma^{-k} f} \Sigma^{-2k} X \to \cdots)$

(2) $\mathbb{Z} = \pi_0(S^0) \to \pi_0 \mathrm{End}(X)$.

S = multiplicative subset of $\pi_* \mathrm{End}(X)$.

$X[S^{-1}] = X[\frac{1}{f} \mid f \in S]$.

$S = \{ n \in \mathbb{Z} \mid (n, p) = 1 \ n \neq 0 \}$.

$X_{(p)} = X[S^{-1}]$

$X_{\mathbb{Q}} = X[(\mathbb{Z} \setminus \{0\})^{-1}]$

(3) $X/f = \mathrm{cofib}(\Sigma^k X \xrightarrow{f} X)$.

$X^{\wedge}_{f} := \mathrm{holim}(X/f \leftarrow X/f^2 \leftarrow \cdots)$.

Examples: (1). $E = M(\mathbb{Z}[\frac{1}{p}])$ p prime.

then $L_E X = X[\frac{1}{p}] \simeq X \wedge S^0[\frac{1}{p}]$.

pf: $X \longrightarrow X[\frac{1}{p}]$ is an E-equiv into E-loc spectrum.

UCT: $\Rightarrow$ $\pi_*(Y \wedge M\mathbb{Z}[\frac{1}{p}]) \simeq \pi_*(Y) \otimes \mathbb{Z}[\frac{1}{p}]$.

$\Rightarrow$ $X \longrightarrow X[\frac{1}{p}]$ is an E-equiv.

Y is E-acyclic $\Leftrightarrow$ $Y \xrightarrow{p} Y$ is zero.

$[Y, X[\frac{1}{p}]] = 0$. $\Rightarrow$ $X[\frac{1}{p}]$ is E-loc.

(2). $E = M(\mathbb{Z}_{(p)})$, then $L_E X \simeq X_{(p)}$
$\simeq X \wedge S^0_{(p)}$

(3) $E = M\mathbb{Q}$, then $L_E X = X_{\mathbb{Q}} \simeq X \wedge S^0_{\mathbb{Q}}$

we write $L_{\mathbb{Q}}$ for this L_E.

(4). $E = M(\mathbb{Z}/p)$, then $L_E X = X^{\wedge}_p$.

NOT smashing! $L_p := L_E$.

Thm (Sullivan arithmetic fracture sq).

$\forall X \in Sp$. we have pull back sq.

$$\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \prod_p L_p X \arrow[d] \\
L_{\mathbb{Q}} X \arrow[r] & L_{\mathbb{Q}}\left(\prod_p L_p X\right).
\end{tikzcd}$$