Chromatic Lecture 13:

§1. Geometry of $\mathcal{M}_{fg}$.
§2. Chromatic localizations.
§3. Monochromatic Layers.
§4. Telescopic localization.
~~§5. Height 1 computations~~

References:

- Ravenel's paper: Localization wrt certain periodic homology thy.
- The orange book.
- Lurie's notes.
- Gross-Hopkins: "The rigid analytic ......"
- Goerss: Qcoh sheaves over $\mathcal{M}_{fg}$.

Recall at end of last lecture

Thm $\forall X \in Sp$, there is a fracture sq:

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \prod_p L_p X \arrow[d] \\ L_{\mathbb{Q}} X \arrow[r] & L_{\mathbb{Q}} \prod_p L_p X \end{tikzcd}$$

$\rightsquigarrow$ We can recover a spectrum X from $L_{\mathbb{Q}} X$ & $L_p X$ at all primes p.

$L_{\mathbb{Q}} X \simeq \prod_n \Sigma^n H(\pi_n(X) \otimes \mathbb{Q})$

However $L_p X = X_p^\wedge$ has even finer structure.

Recall a FGL over $\mathbb{Q}$ is isom to $\mathbb{G}_a$.

$\rightsquigarrow \quad Sp \longrightarrow \mathcal{D}\,QCoh(\mathcal{M}_{fg})$.

moduli stack of Formal Group

§1. Geometry of $\mathcal{M}_{fg}$

Recall $\forall X \in Sp$, there is an Adam-Novikov spectral sequence:

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{MU_*MU}(MU_*, MU_*(X)) \Rightarrow \pi_{t-s}(X).$$

classifies FGL.

Quillen's thm:

$$\underset{\text{Hopf algebroid}}{(MU_*, MU_*MU)} \cong (\underset{\uparrow}{L}, \underset{\uparrow}{W})$$

classifies strict isoms b/t FGL.

$\mathrm{Spec}\, L // \mathrm{Spec}\, W =$ moduli stack of formal gps. $\mathcal{M}_{fg}$.

$$\{MU_*MU\text{-comods}\} \simeq \mathrm{Qcoh}(\mathcal{M}_{fg}).$$

$$MU_*(X) \rightsquigarrow \mathcal{F}_X$$

$$E_2^{s,t} \cong H^s(\mathcal{M}_{fg}, \mathcal{F}_X \otimes \omega^{\otimes t/2})$$

$$Sp \xrightarrow{MU_*(-)} \mathcal{D}\, \mathrm{Qcoh}(\mathcal{M}_{fg}).$$

Slogan: The geometry of $\mathcal{M}_{fg}$ reflects the structure Sp.

$(\mathcal{M}_{fg})_{\mathbb{Q}}$ has 1 pt b/c FGLs over $\mathbb{Q}$-alg are isom to $\mathbb{G}_a$.

$(\mathcal{M}_{fg})^\wedge_p$ has finer structure:

Recall: Over $\mathbb{F}_p^{sep}$, FGLs are classified by their heights.

$\mathcal{M}_{fg} \otimes \overline{\mathbb{F}_p}$ = has 1 "pt" for each ht.

$$(\mathcal{M}_{fg}) \otimes \mathbb{F}_p \supseteq \cdots\cdots \supseteq \underset{\substack{\wr\\ ht \le 2}}{\mathcal{U}_2} \supseteq \underset{\substack{\wr\\ ht \le 1}}{\mathcal{U}_1} \supseteq \underset{\substack{\wr\\ ht \le 0}}{\mathcal{U}_0}$$

$\mathcal{U}_n \setminus \mathcal{U}_{n-1} = \mathcal{H}_n$ classifies FGs of ht exact n

$= \operatorname{Spec} \mathbb{F}_{p^n} /\!/ \mathbb{G}_n$.

Formal nbhd of $\mathcal{H}_n \subseteq \mathcal{U}_n$: Morava E-th.

$$\widehat{\mathcal{H}_n} = \operatorname{Spec} \pi_0 E_n /\!/ \mathbb{G}_n$$

$(\mathcal{M}_{fg})^\wedge_p \longleftrightarrow$ p-complete Spectra

$\mathcal{U}_n$ ---- ?

$\mathcal{H}(n)$ ---- $\sim$??

§2. Chromatic Localizations.

Goal: Want to study Bousfield localization wrt Morava E-thy & Morava K-theory.

Defn. Two spectra X_1, X_2 are Bousfield equivalent if one of the following equivalent conditions hold.

- $L_{X_1} \simeq L_{X_2}$.
- E is X_1-local $\Leftrightarrow$ E is X_2-local.

  $Sp_{X_1} \rightleftarrows Sp_{X_2}$.
- E is X_1-acyclic $\Leftrightarrow$ E is X_2-acyclic

In this case, we will write $\langle X_1 \rangle = \langle X_2 \rangle$

Thm: $\langle E_n \rangle = \langle E_{n-1} \vee K(n) \rangle$

$$= \langle E_{n-1} \rangle \vee \langle K(n) \rangle$$

Induction $= \langle K(0) \rangle \vee \langle K(1) \rangle \cdots \vee \langle K(n) \rangle$

$K(0) = H\mathbb{Q}$

Lem: E = ring spectrum. $v \in \pi_k E$.

then $\langle E \rangle = \langle v^{-1} E \rangle \vee \langle E/v \rangle$.

$v : S^k \to E$.

"v" : $\Sigma^k E \xrightarrow{v \wedge 1} E \wedge E \to E$. $|u| = 2$

Example: $\pi_* E_1 = \underbrace{\mathbb{Z}_p[[u_1]]}_{\deg = 0} [u^{\pm 1}]$

$v = p \in \pi_0 E_1$. $\mathbb{F}_p[[u_1]][u^{\pm}]$

$\Rightarrow \langle E_1 \rangle = \langle p^{-1} E \rangle \vee \langle E/p \rangle$.

rational

$= \langle K(0) \rangle \vee \langle K(1) \rangle$.

$K(0) = M\mathbb{Q}$ $\quad K(1)_* = \mathbb{F}_p[v^{\pm}]$

Lem: $E, F, X \in Sp$, s.t. $L_F X$ is E-acyclic

$$\Rightarrow \begin{tikzcd} L_{E \vee F} X \arrow[r] \arrow[d] \arrow[rd, phantom, "\sim"] & L_E X \arrow[d] \\ L_F X \arrow[r] & L_F L_E X \end{tikzcd}$$?

A spectrum Y is $E \vee F$ acyclic.

$\rightsquigarrow (E\vee F)_* Y = 0 \iff (E)_*(Y) \oplus (F)_*(Y) = 0$

$\iff Y$ is both E-acyclic

& F-acyclic.

If Y is E-local, then it is automatically $E \vee F$-local.

$\Rightarrow L_{E\vee F} X$ is not E-local or F-local.

$L_E L_{E \vee F} \simeq L_E$.

Take $E = K(n)$. $F = E_{n-1}$.

Can check: E_{n-1}-local spectra are all $K(n)$-local.

$\rightsquigarrow \forall X$, we have a pull back sq.

Denote L_{E_n} by L_n.

$$\begin{tikzcd} L_n X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & L_{K(n)} X \arrow[d] \\ L_{n-1} X \arrow[r] & L_{n-1} L_{K(n)} X \end{tikzcd}$$

$L_{n-1} L_n \simeq L_{n-1}$.

Thm (chromatic convergence).

$X_{(p)} \simeq \operatorname{holim}(\to \cdots \to L_n X \to L_{n-1} X \to \cdots \to L_0 X)$.

Now: We can try to recover $X_{(p)}$ from $L_n X$.

Fracture square: we can recover $L_n X$ from $L_{K(0)} X, \cdots, L_{K(n)} X$.

We can compute $\pi_* L_{K(n)} X$ using the homotopy fixed pt s.s.

Fact: $L_{K(n)} X \simeq [L_{K(n)}(E_n \wedge X)]^{hG_n}$.

$\rightsquigarrow H_c^s(G_n; (E_n^\wedge)_t(X)) \Rightarrow \pi_{t-s} L_{K(n)} X$.

can think as a sheaf cohomology over $\widehat{\mathcal{M}(n)}$.

§3. Monochromatic layer

BP_* - ANSS.

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*) \Rightarrow \pi_{t-s}(S^0_{(p)}).$$

$s=0$ line $\mathrm{Ext}^{0,t}_{BP_*BP}(BP_*, M)$ is easier to compute.

One way to compute $E_2^{s,t}$ is.

set. $N^0 = BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots, v_n, \dots]$

$M^k = v_k^{-1} N^k$. $\quad p = v_0$.

$0 \to N^k \to M^k \to N^{k+1} \to 0$.

$M^0 = \mathbb{Q}[v_1, \dots v_n \dots$

$N^1 = \mathbb{Z}/p^\infty [v_1, \dots v_n \dots$

$M^1 = v_1^{-1} \mathbb{Z}/p^\infty [v_1, v_2 \dots, v_n \dots$

Chromatic S.S.

$$\mathrm{Ext}^{r,s}_{BP_*BP}(BP_*, M^t) \Rightarrow \mathrm{Ext}^{r+t,s}_{BP_*BP}(BP_*, BP_*)$$

Q: Can we realize this construction topologically?

A: Yes.

Fact: If BP_*X is $(p, v_1, \cdots v_{n-1})$ torsion,

then $BP_* L_n X = v_n^{-1} BP_* X$.

Construction: $X \in Sp$,

set $N^0 X = X$

$M^k X = L_k N^k X$

$$N^k X \to M^k X \to N^{k+1} X$$

Thm • when $X = S^0$, BP_*-homology of is the same as the chromatic resolution of BP_*.

• Moreover we can identify $M^k X$

$M_k X$ $N^k X$ as fibers of localizations.

$$= \Sigma^{-k} M^k X \to L_k X \to L_{k-1} X$$

$$\Sigma^{-k} N^k X \to X \to L_{k-1} X$$

$M_k X$ is the k-th monochromatic layer of X

Q: How do $M_n X$ & $L_{K(n)} X$ compare?

A: They determine each other.

- $L_{K(n)} M_n X \simeq L_{K(n)} X$
- $M_n L_{K(n)} X \simeq M_n X$

§4. Telescope localization.

Another way to extract ht n information is by localization wrt finite complexes of type n.

Fact: Any two finite complexes $F(n)$ of type n are Bousfield equivalent.

Let $v_n : F(n) \to F(n)$ self map.

$T(n) = v_n^{-1} F(n)$

Telescopic localization is $L_{T(n)}$.

- $L_n^f := L_{T(0) \vee T(1) \vee \cdots \vee T(n)}$.

Q: How to compare $L_{K(n)}$ & $T(n)$.

A: $Sp_{K(n)} \subseteq Sp_{T(n)}$.

$$\rightsquigarrow L_{T(n)}X \longrightarrow L_{K(n)}X.$$

Telescope conjecture:

$$L_{T(n)}X \xrightarrow{\sim} L_{K(n)}X.$$

equivalently. $L_n^f X \simeq L_n X$.

Only $n=1$ case has been proved.