iWoAT Summer School on Chromatic Homotopy Theory and Higher (Infinity-Categorical) Algebra

L2: 1-category theory of ∞-categories, I

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Acknowledgement !

- I will follow the first half of the lecture notes by my advisor Charles, available at

  faculty.math.illinois.edu/~ rezk/quasicats.pdf

  Private conversations with Charles is also a major reference for my talks.

- Special thanks to David Gepner for coordinating his talk with mine and offering to cover some of the more challenging topics that technically fall under my talks.

- I would like to thank also

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  for very helpful conversations during the preparation of my lectures.

Q: Why work in an ∞-setting? $\boxed{\infty\text{-} = (\infty,1)\text{-}}$

"A": It's the right place to do (generalized) homotopy theory using (generalized) topological spaces.

Solutions to a problem up to some equivalence are organized in the form of a groupoid.

e.g. X a top. space $\rightsquigarrow$ F sheaf on X valued in groupoids

for U, V open in X

$$\begin{tikzcd} F(U \cup V) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & F(V) \arrow[d] \\ F(U) \arrow[r] & F(U \cap V) \end{tikzcd}$$

weak pullback in groupoids

$\rightsquigarrow$ need category of groupoids & its homotopy theory.

Better: ∞-category of ∞-groupoids.

$\hookrightarrow$ higher groupoids: solve higher problems

"homotopy hypothesis" $1 \leq n \leq \infty$

$$(n\text{-groupoids}) \Leftrightarrow (n\text{-truncated spaces})$$

$$n = 1 \Rightarrow \quad \pi_1(BG) \simeq G, \quad \pi_{k>1}(BG) \simeq 0$$

$\hookrightarrow$ ∞-categorical: better colimits/limits (as opposed to say, derived functor in QMC)

$\rightsquigarrow$ ∞-category of ∞-sheaves valued in ∞-groupoids ...

∞-groupoid = $(\infty, 0)$-category
morphisms of dim > 0 are invertible

$\rightsquigarrow$ $(\infty,1)$-category, which we will model by a quasi-category "qcat".

L2 & L4: understanding qCat.

- In this model, a qcat is a particular kind of simplicial sets
- We will verify the homotopy hypothesis in this model.
- In general, qcats give a favorite model for ∞-cats; but there are settings where others are nicer. David will talk about one such model tomorrow.
- I will use "qcat" & "∞-cat" synonymously.

Glossary

$$\left.\begin{array}{l}
\mathrm{Cat} = 1\text{-category of categories} \\
\mathrm{Cat}_1 = \infty\text{-category of categories} \\
q\mathrm{Cat} = 1\text{-category of } \infty\text{-categories} \\
s\mathrm{Set} = 1\text{-category of simplicial sets}
\end{array}\right\} \text{Today!}$$

Cat_∞ = ∞-category of ∞-categories (David, L6 & L9)

Simplicial operator category Δ

obj: $[n] := \{0 < 1 < 2 < \cdots < n\}$ for $n \geq 0$

mor: $f : [m] \to [n]$ monotone

"simplicial operators"

$\hookrightarrow$ face operators $d^i = \langle 0, \dots, i-1, \hat{i}, i+1, \dots n \rangle : [n-1] \to [n]$

degeneracy operators $s^i = \langle 0, \dots, i-1, i, i, i+1, \dots n \rangle : [n+1] \to [n]$

Simplicial Set functor $X : \Delta^{op} \to Set$

$\to$ $X_n := X[n]$, the set of n-cells/n-simplices in X

$a \in X_m$ is degenerate if $a = bf$ for some cell $b \in X$ and f non-injective simplicial operator

$a \in X_m$ is nondegenerate otherwise.

$\hookrightarrow$ $X_n^{deg} = \{ af \mid a \in X_k,\ f : [n] \to [k],\ k < n \}$

$\to$ sSet := 1-category of simplicial sets

mor: natural transformations of functors.

Set_Δ in kerodon, §2.4.2

$\to$ Standard n-simplex $\Delta^n = Hom_\Delta(-, [n]) \in sSet$

$\Delta^n_m = Hom_\Delta([m], [n]) = \{ f : [m] \to [n] \}$

Yoneda lemma $Hom_{sSet}(\Delta^n, X) \to X_n$

$g \mapsto g(id_{[n]})$

Rmk. Yoneda: d^i induces $(d_i)_* : \Delta^{n-1} \to \Delta^n$ as you learned in a standard first course in AT: $(d_i)_* : \Delta^{n-1}_{top} \to \Delta^n_{top}$.

→ pictures of $\Delta^n = \mathrm{Hom}_\Delta(-,[n])$

$$\Delta^n_m = \mathrm{Hom}_\Delta([m],[n]) = \{ f : [m] \to [n] \}$$

Δ^0:

$$\langle 0 \rangle$$

Δ^1:

$$\langle 0 \rangle \to \langle 1 \rangle$$

Δ^2:

$$\begin{tikzcd} & \langle 1 \rangle \arrow[dr] & \\ \langle 0 \rangle \arrow[ur] \arrow[rr] \arrow[rr, phantom, shift left=4ex, "\alpha"] & & \langle 2 \rangle \end{tikzcd}$$

Δ^3:

$$\begin{tikzcd} & \langle 1 \rangle \arrow[rr] \arrow[dr] & & \langle 3 \rangle \\ \langle 0 \rangle \arrow[ur] \arrow[rr] \arrow[urrr] & & \langle 2 \rangle \arrow[ur] & \end{tikzcd}$$

only drawing non-deg. cells

sSet has small (co)limits and computed degreewise.

→ initial obj: empty simplicial set $[n] \mapsto \emptyset$

terminal obj: $\Delta^0 = *$

Subcomplexes

→ <u>subcomplex</u> A of X is just a subfunctor.

$A_n \subseteq X_n$ are closed under the action of simplicial operators.

- every subcomplex K of Δ^n is a colimit:

  $\mathcal{A} :=$ poset of all nonempty $S \subseteq [n]$ s.t. $\Delta^S \subseteq K$

  $\Rightarrow \operatorname{colim}_{S \in \mathcal{A}} \Delta^S \xrightarrow{\simeq} K$ isomorphism.

- if K, L are subcomplexes of $X \in \mathrm{sSet}$,

  $K \cup L = \mathrm{Colim}(K \leftarrow K \cap L \rightarrow L)$, $\mathcal{A} = \{K, L, K \cap L\}$

  pushout of simplicial sets.

→ (co)limits in sSet

For any $F : C \longrightarrow \mathrm{Set}$, C small cat, define "$\sim$" on $\coprod_{c \in \mathrm{ob}\, C} F(c)$ by

$$(c, x) \sim (c', x') \text{ if } \exists\, \alpha : c \to c' \text{ in } \mathrm{mor}\, C \text{ s.t. } F(\alpha)(x) = x'.$$

not symm.

Let $X := \coprod_{c \in \mathrm{ob}\, C} F(c) / \approx$, where $\approx$ is generated by $\sim$.

For $c \in C$, have $\iota_c : F(c) \longrightarrow X$, $x \longmapsto [(c, x)]$

Then $(X, \{\iota_c\})$ is a colimit of F.

A colimit of $J : C \longrightarrow s\mathrm{Set}$ in simplicial sets is computed degreewise, i.e.

$$\mathrm{colim}\, J = \mathrm{colim}_{c \in C} J(c) \Rightarrow (\mathrm{colim}\, J)_n = \mathrm{colim}_{c \in C} J(c)_n$$

When a colimit of functors to Set is more trackable ...

Prop. Let $\mathcal{A}$ be a collection of subsets of a set S, regarded as a poset under "$\subseteq$". If for any $s \in S$, $T, U \in \mathcal{A}$ s.t. $s \in T \cap U$, there exists $V \in S$, with $s \in V \subseteq T \cap U$, Then

$$\begin{array}{ccc} \mathrm{colim}_{T \in \mathcal{A}}\, T & \longrightarrow & \bigcup_{T \in \mathcal{A}} T \\ [(T, t)] & \longmapsto & t \end{array}$$

is a bijection.

Rmk. $\mathcal{A}$ satisfies a weaker closed under intersection condition.

For $X, Y \in sSet$,

product "$\times$" and coproduct "$\amalg$" are defined degreewise

- $(X \times Y)_n = X_n \times Y_n$; $X \times Y \in sSet$ (easy)
- $(X \amalg Y)_n = X_n \amalg Y_n$; $X \amalg Y \in sSet$? (yes.)
- Connected component $\pi_0 X$ of X

  "$\approx$" equivalence relation on $\coprod_{n \geq 0} X_n$ gen. by

  $a \sim af$ for all $n \geq 0$, $a \in X_n$, $f : [m] \longrightarrow [n]$

  $\pi_0 X = \{$equivalence classes for $\approx\}$

$\rightsquigarrow$ $\pi_0 : sSet \longrightarrow Set$ is well-defined & monoidal w.r.t. "$\times$".

↳ Each connected component is a subcpx and $\coprod_{C \in \pi_0 X} C \xrightarrow{\simeq} X$;

↳ For any n, Δ^n are connected, i.e. π_0 is a singleton.

↪ A coproduct $X = \coprod_s X_s$, $X_s \in sSet$ is a simplicial set.

→ (Cartesian) product of $X, Y \in sSet$

$$(X \times Y)_n = X_n \times Y_n \qquad \text{Cartesian product in Set}$$

$$f : [m] \longrightarrow [n] \quad \text{acts by} \quad (a,b) = (af, bf)$$

Example. $X = \Delta^1 \times \Delta^1$

Let A, B be subcollection of cells in X s.t.

$$(f,g) \in \begin{cases} A & \text{if } f(i) \leq g(i), \text{ all } 0 \leq i \leq n \\ B & \text{if } f(i) \geq g(i), \text{ all } 0 \leq i \leq n \end{cases}$$

for simplicial operators $f, g : [n] \longrightarrow [1]$.

Then $A \approx \Delta^2 \approx B$, $A \cap B \approx \Delta^1$ as subcpxes of X.

$$\begin{tikzcd}
(0,1) \arrow[r] & (1,1) \\
(0,0) \arrow[r] \arrow[u] \arrow[ur, "A", "B"'] & (1,0) \arrow[u]
\end{tikzcd}$$

In fact,

$$\begin{tikzcd}
\Delta^1 \arrow[r, "\langle 02 \rangle"] \arrow[d, "\langle 02 \rangle"'] & \Delta^2 \arrow[d, "{\langle 011 \rangle, \langle 001 \rangle}"] \\
\Delta^2 \arrow[r, "{\langle 001 \rangle, \langle 011 \rangle}"'] \arrow[ur, phantom, "\ulcorner"] & \Delta^1 \times \Delta^1
\end{tikzcd}$$

L4: This is a colimit in ∞-cats!

Horns For $n \geq 1$, $\Lambda^n_j \subseteq \Delta^n$ subobject for $0 \leq j \leq n$

$$(\Lambda^n_j)_k = \{ f : [k] \longrightarrow [n] \mid ([n] \setminus \{j\}) \not\subseteq f(k) \}$$

$$\Rightarrow \quad \boxed{\Lambda^n_j = \bigcup_{i \neq j} \Delta^{n \setminus \{i\}} \subseteq \Delta^n}$$

is the subcpx obtained by removing the non-deg n-cell and the face (codim 1 cell) opposite of j.

$\rightarrow$ $j < n$ <u>Left horn</u> $\qquad 0 < j < n$ <u>Inner horn</u>

$0 < j$ <u>right horn</u>

$\rightarrow$ Pictures: $\Lambda^1_0 = \Delta^{\{0\}}$ $\qquad \Lambda^1_1 = \Delta^{\{1\}}$ $\qquad \Delta^1$:

$\langle 0 \rangle \qquad\qquad \langle 1 \rangle \qquad\qquad \langle 0 \rangle \longrightarrow \langle 1 \rangle$

Λ^2_0

$$\begin{tikzcd} & \langle 1 \rangle & \\ \langle 0 \rangle \arrow[ur, "\langle 01 \rangle"] \arrow[rr, "\langle 02 \rangle"] & & \langle 2 \rangle \end{tikzcd}$$

left horn

Λ^2_1

$$\begin{tikzcd} & \langle 1 \rangle \arrow[dr, "\langle 12 \rangle"] & \\ \langle 0 \rangle \arrow[ur, "\langle 01 \rangle"] & & \langle 2 \rangle \end{tikzcd}$$

Λ^2_2

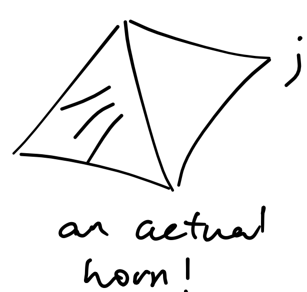
$$\begin{tikzcd} & \langle 1 \rangle \arrow[dr, "\langle 12 \rangle"] & \\ \langle 0 \rangle \arrow[rr, "\langle 02 \rangle"] & & \langle 2 \rangle \end{tikzcd}$$

right horn

Δ^2

$$\begin{tikzcd} & \langle 1 \rangle \arrow[dr] \arrow[d, phantom, "\alpha"] & \\ \langle 0 \rangle \arrow[ur] \arrow[rr] & {} & \langle 2 \rangle \end{tikzcd}$$

Λ^3_j



an actual horn!

Nerve : • For any $C \in Cat$, $NC \in sSet$ s.t.

$$NC_n = Hom_{Cat}([n], C)$$

• $F: C \to D$ functor, $NF : NC \to ND$

$$(\alpha : [n] \to C) \longmapsto (F\alpha : [n] \to D)$$

$\Rightarrow$ $N(-) : Cat \longrightarrow sSet$

$\rightarrow$ $N[m] = \Delta^m$ as sSet.

$\rightarrow$ $NC_0 = ob(C)$,

$\rightarrow$ $NC_1 = mor(C)$

$\langle 0 \rangle^*, \langle 1 \rangle^* : NC_1 \to NC_0$, mor $\longmapsto$ source, taget

$\langle 0\,0 \rangle^* : NC_0 \longrightarrow NC_1$, $x \longmapsto id_x$

$\rightarrow$ $NC_2 \approx \{$ pairs of composable morphisms $\}$

$$NC_2 \longrightarrow NC_1 \times NC_1$$
$$a \longmapsto (a_{01}, a_{12})$$

$$\begin{tikzcd} & 1 \arrow[dr, "g"] \arrow[d, phantom, "a"] & \\ 0 \arrow[ur, "f"] \arrow[rr, "gf"'] & {} & 2 \end{tikzcd} \longmapsto (g, f)$$

Characterization of Nerves.

Prop. $X \in sSet$ is isomorphic to the nerve of some category iff for all $n \geq 2$, the following function

$$\phi_n : X_n \longrightarrow \{(g_1, \cdots, g_n) \in (X_1)^{\times n} \mid g_{i-1}\langle 1\rangle = g_i\langle 0\rangle \text{ for all } 1 \leq i \leq n\}$$

$$a \longmapsto (a_{01}, \cdots, a_{n-1,n})$$

$a_{ij} = a \circ \langle ij\rangle$

is a bijection.

Cor. Maps to nerves are determined by edges.

Maps between Nerves.

aka. why we identify a 1-cat with its nerve in sSet.

Thm. $N : Cat \to sSet$ is fully faithful, i.e.

$$\mathrm{Hom}_{Cat}(C, D) \cong \mathrm{Hom}_{sSet}(NC, ND).$$

Proof: injectivity is clear since a functor (between cats) is determined by its value on objs & mors.

surjectivity: given $g : NC \to ND$, g_0, g_1 gives value of functor f on objs & mors;

The composition of a functor is unique by the previous prop:

Proof continued: $f(\mathrm{id}_x) = \mathrm{id}_{f(x)}$ is $g\langle 00\rangle^* = \langle 00\rangle^* g$

$$
\begin{tikzcd}
NC_0 \arrow[r, "\langle 00\rangle^*"] \arrow[d, "g_0"] & NC_1 \arrow[d, "g_1"] \\
ND_0 \arrow[r, "\langle 00\rangle^*"] & ND_1
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r, mapsto] \arrow[d, mapsto] & \mathrm{id}_x \arrow[d, mapsto] \\
f(x) \arrow[r, mapsto] & f(\mathrm{id}_x) = \mathrm{id}_{f(x)}
\end{tikzcd}
$$

$$f(x \to y \to z) = f(y \to z)\, f(x \to y)$$

$$
\begin{tikzcd}
NC_2 \arrow[r, "\langle 02\rangle^*"] \arrow[d, "g_2"] & NC_1 \arrow[d, "g_1"] \\
ND_2 \arrow[r, "\langle 02\rangle^*"] & ND_1
\end{tikzcd}
\qquad
\begin{tikzcd}
& y \arrow[dr] & \\
x \arrow[ur] \arrow[rr] \arrow[r, phantom, "a"] & & z
\end{tikzcd}
\longmapsto
\begin{array}{c} x \to z \\ \| \\ x \to y \to z \end{array}
$$

$$
\downarrow \qquad\qquad \downarrow
$$

$$
\begin{tikzcd}
& \cdot \arrow[dr, "f(y\to z)"] & \\
\cdot \arrow[ur, "f(x\to y)"] \arrow[rr] \arrow[r, phantom, "g(a)"] & & \cdot
\end{tikzcd}
\longmapsto
\begin{array}{c} f(x \to y \to z) \\ \| \\ f(y \to z)\, f(x \to y) \end{array}
$$

let a be a 2-cell in NC_2 s.t. $a_{01} = x \to y$, $a_{12} = y \to z$. Then by previous prop, a is uniquely determined by the composable pair $(x \to y, y \to z)$, and must have

$$a_{02} = x \to y \to z.$$

Similarly, the composable pair $(f(x \to y), f(y \to z))$ uniquely determines a 2-cell in ND_2, which must be $g(a)$ since

$$g(a)_{01} = f(x \to y), \quad g(a)_{12} = f(y \to z).$$

Thus

$$g(a)_{02} = f(y \to z)\, f(x \to y)$$

uniquely determines the composition rule of f.

Uniqueness of f follows from previous corollary. $\square$

∞-cat / quasicat C is a simplicial set satisfying the inner horn extension property for all $n \geq 2$ and $0 < j < n$.

$$\begin{tikzcd} \Lambda^n_j \arrow[r] \arrow[d, hook] & C \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

Prop. A simplicial set X is isomorphic to the nerve of a cat iff it has the unique inner horn extension property.

$$X_n \cong \mathrm{Hom}_{sSet}(\Delta^n, X) \xrightarrow{\cong} \mathrm{Hom}_{sSet}(\Lambda^n_j, X)$$

So $N : \mathrm{Cat} \longrightarrow q\mathrm{Cat} \underset{\text{full}}{\subseteq} s\mathrm{Set}$

Remark:

- "Let C be an ∞-category, which is a category" = "Let C be an ∞-category isomorphic to nerve of some cat".
- Composition in an ∞-cat cannot be defined the same as for a 1-cat.

Convention: For C an ∞-cat: C_0 = objs, C_1 = mors;
for $x \in C_0$, $id_x = x\langle 00 \rangle \in C_1$;
for $f \in C_1$, $f\langle 0 \rangle$ = source, $f\langle 1 \rangle$ = target.

proof of prop via:

$$\mathrm{Hom}_{sSet}(\Delta^n, X) \overset{\alpha}{\simeq} X_n \longrightarrow \mathrm{Hom}_{sSet}(\Lambda^n_j, X) \xrightarrow{r} \mathrm{Hom}_{sSet}(I^n, X)$$

"only if": • r is injective : straightforward.

• composite is bijective :

I^n = "spine" = longest chain of edges in Δ^n

$$I^3: \begin{tikzcd} & \langle 1 \rangle \arrow[dr] & & \langle 3 \rangle \\ \langle 0 \rangle \arrow[ur] & & \langle 2 \rangle \arrow[ur] & \end{tikzcd} \qquad \Delta^3: \begin{tikzcd} & \langle 1 \rangle \arrow[dr] \arrow[rr] & & \langle 3 \rangle \\ \langle 0 \rangle \arrow[ur] \arrow[urrr] \arrow[rr] & & \langle 2 \rangle \arrow[ur] & \end{tikzcd}$$

"if": • composition rule on X_1

$$\begin{tikzcd} \Lambda^2_1 \arrow[r, "{(f,g)}"] \arrow[d] & X \\ \Delta^2 \arrow[ur, dashed, "\exists!\, u"'] & \end{tikzcd}$$

$u_{02} = gf$

• composition is unital if we let one of f, g to be id.

• composition is associative:

$$\begin{tikzcd} & 0 \arrow[d, "f"] \arrow[dddl, bend right, "gf"'] \arrow[ddr, bend left, "(hg)f"] & \\ & & \\ & 1 \arrow[r, "hg"] \arrow[dl, "g"] & 3 \\ 2 \arrow[urr, bend right, "h"'] & & \end{tikzcd}$$

$\Lambda^3_1 \to X$ extends uniquely to $v : \Delta^3 \to X$ with $v\langle 023 \rangle$ witnessing $h(gf) = (hg)f$

$\rightsquigarrow$ can define $\alpha_n : X_n \to NC_n$ by restricting to I_n

$\rightarrow$ bijective: obvious for $n \leq 1$ + induction. $\square$

Opposite of an ∞-cat.

Given $C \in Cat$, C^{op}: $\text{obj } C^{op} = \text{obj } C$,

$$Hom_{C^{op}}(X, Y) = Hom_C(Y, X)$$

For $X \in sSet$, $X^{op} = X \circ op \rightsquigarrow (\Delta^n)^{op} \simeq \Delta^n$, $(\Lambda^n_j)^{op} \simeq \Lambda^n_{n-j}$

For $X \in \infty\text{-Cat}$, $X^{op} \in \infty\text{-Cat}$; $(NC)^{op} = N(C^{op})$

$op \circ op = id \Rightarrow (X^{op})^{op} = X$.

Functor between ∞-cats is just a functor between simplicial sets.

Nat trans $h: f_0 \Rightarrow f_1$ between $f_0, f_1 : C \to D$ in $qCat$, is the map

$$h: C \times N[1] = C \times \Delta^1 \longrightarrow D \quad \text{in } sSet$$

s.t. $h|C \times \{i\} = f_i$.

Subcats of an ∞-cat. A subcpx $C' \subseteq C$ for $C \in qCat$ is a subcategory if for $n \geq 2$, $0 < k < n$, every $f: \Delta^n \to C$ s.t. $f(\Lambda^n_k) \subseteq C'$ satisfies $f(\Delta^n) \subseteq C'$.

It is a **full subcat** if for all n and all $a \in C_n$,

$$a \in C_n' \Leftrightarrow a_i \in C_0' \text{ for all } 0 \leq i \leq n.$$

Warning: If C', C are ∞-cats and $C' \subseteq C$ is a subcpx, i.e. a subfunctor, C' need NOT be a subcat of C as ∞-cats.

Cat_1 := ∞-category of small 1-cats.

objs / 0-cells : small cats

1-morphisms / 1-cells : functors

2-morphisms / 2-cells : nat isos $\zeta_{012} : F_{02} \Rightarrow F_{12}F_{01}$.

$(\mathrm{Cat}_1)_n := \{ (C_i, F_{ij}, \zeta_{ijk})$.

(0) for each $i \in [n]$, C_i is a small category.

(1) for each $i \le j$ in $[n]$, $F_{ij} : C_i \longrightarrow C_j$ is a functor.

(2) for each $i \le j \le k$ in $[n]$, $\zeta_{ijk} : F_{ik} \Rightarrow F_{jk}F_{ij}$ is a nat isomorphism of functors $C_i \to C_k$

s.t.
- for each $i \in [n]$, F_{ii} is id_{C_i}.
- for each $i \le k$ in $[n]$, $\zeta_{iik} : F_{ik} \Rightarrow F_{ik}\,\mathrm{id}_{C_i}$, $\zeta_{ikk} : F_{ik} \Rightarrow \mathrm{id}_{C_k}F_{ik}$ are identity nat transformation of F_{ik}.
- for each $i \le j \le k$ in $[n]$, the diagram commutes

$$\begin{tikzcd}
F_{i\ell} \arrow[r, Rightarrow, "\zeta_{ij\ell}"] \arrow[d, Rightarrow, "\zeta_{ik\ell}"'] \arrow[dr, phantom, "\circlearrowright"] & F_{j\ell}F_{ij} \arrow[d, Rightarrow, "\zeta_{jk\ell}F_{ij}"] \\
F_{k\ell}F_{ik} \arrow[r, Rightarrow, "F_{k\ell}\zeta_{ijk}"'] & F_{k\ell}F_{jk}F_{ij}
\end{tikzcd}$$

$\zeta_{jk\ell}F_{ij} \circ \zeta_{ij\ell} = F_{k\ell}\zeta_{ijk} \circ \zeta_{ik\ell}$ as nat isos. $\}$

$\delta : [m] \longrightarrow [n]$ acts by

$$\delta(C_i, F_{ij}, \zeta_{ijk}) = (C_{\delta(i)}, F_{\delta(i)\delta(j)}, \zeta_{\delta(i)\delta(j)\delta(k)})$$

$\mathrm{Cat}_1 \in q\mathrm{Cat}$:

- Given $\Lambda^2_1 \to \mathrm{Cat}_1$,

$$\begin{tikzcd} & C_1 \arrow[dr, "F_{12}"] & \\ C_0 \arrow[ur, "F_{01}"] & & C_2 \end{tikzcd}$$

we can extend it to $\zeta_{012} = id_{F_{12}\circ F_{01}}$, $F_{02} = F_{12}\circ F_{01}$.

But this filler is not unique:

- 2-cells in Cat_1 are nat isos $\zeta_{012} : F_{02} \Rightarrow F_{12}F_{01}$.

So we just need F_{02} to be nat isomorphic to $F_{12}\circ F_{01}$.

- Extensions for higher inner horns are unique.

filler for Λ^3_1 is the commutative diagram with the filled-in face ζ_{023}.

$$\begin{tikzcd} & C_1 \arrow[rr] \arrow[dr] & & C_3 \\ C_0 \arrow[ur] \arrow[rr] \arrow[urrr] & & C_2 \arrow[ur] & \end{tikzcd}$$

$$\begin{tikzcd} F_{03} \arrow[r, Rightarrow, "\zeta_{013}"] \arrow[d, Rightarrow, "\zeta_{023}"'] \arrow[dr, phantom, "\circlearrowright"] & F_{13}F_{01} \arrow[d, Rightarrow, "\zeta_{123}F_{01}"] \\ F_{23}F_{02} \arrow[r, Rightarrow, "F_{23}\zeta_{012}"'] & F_{23}F_{12}F_{01} \end{tikzcd}$$

Fact: $N\mathrm{Cat}$ is isomorphic to the subcpx of Cat_1 consisting of cells $(C_i, F_{ij}, \zeta_{ijk})$ s.t. $F_{ik} = F_{jk}F_{ij}$.

So $N\mathrm{Cat}$ is not a subcategory of Cat_1 since it misses all nat isos $F_{02} \Rightarrow F_{12}\circ F_{01}$ which are not identity nat trans.

Fundamental category for $X \in sSet$ consists of :

1) a category hX

2) $\alpha : X \longrightarrow N(hX)$ in $sSet$ s.t. for any $C \in Cat$

$$\alpha^* : Hom(N(hX), NC) \longrightarrow Hom(X, NC)$$

is a bijection.

Prop : Every simplicial set has a fundamental category.

$$\Rightarrow \quad h : sSet \underset{N}{\overset{\perp}{\rightleftarrows}} Cat : N$$

<u>homotopy relation on morphisms C_1</u> for $C \in qCat$.

$\sim_l$, $\sim_r$ on $hom_C(x,y) = \{ f \in C_1 : f_0 = x, f_1 = y \}$

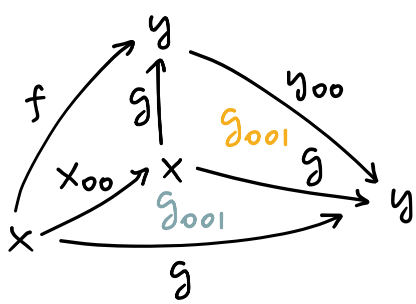
$f \sim_l g \iff \exists a \in C_2, \ a_{01} = 1_x, \ a_{02} = f, \ a_{12} = g$

$f \sim_r g \iff \exists b \in C_2, \ b_{12} = 1_x, \ b_{01} = f, \ b_{02} = g$

$f \sim_l g$:
$$\begin{tikzcd} & x \arrow[dr, "g"] & \\ x \arrow[ur, "x_{00}"] \arrow[rr, "f"'] & a & y \end{tikzcd}$$

$f \sim_r g$:
$$\begin{tikzcd} & y \arrow[dr, "y_{00}"] & \\ x \arrow[ur, "f"] \arrow[rr, "g"'] & b & y \end{tikzcd}$$

- $g \sim_l g$, $g \sim_r g$
- $f \sim_l g \underset{\Delta^{\{0,2,3\}}}{\Rightarrow} f \sim_r g$



$\Lambda^3_1 \longrightarrow C$

- $f \sim_l g, \ g \sim_l h \Rightarrow f \sim_l h$

$\hookrightarrow$ $\sim_l$ and $\sim_r$ are equivalent to each other and are an equivalence relation on $hom_C(x,y)$.

For $f \in \hom_C(x,y)$, $g \in \hom_C(y,z)$, $h \in \hom_C(x,z)$,

h is a composite of (g,f) if $\exists$ 2-cell $a \in C_2$ s.t. $a_{01} = f$, $a_{12} = g$, $a_{02} = h$.

Write $[g] \circ [f]$ for the equivalence class containing a composite of (g,f).

$\to$ It is well-defined, unital & associative.

$\to$ $[h] = [g] \circ [f]$ iff $\exists\, u \in C_2$ s.t. $u|_{\Delta^{\{0,1\}}} = f$, $u|_{\Delta^{\{1,2\}}} = g$, and $u|_{\Delta^{\{0,2\}}} = h$. So every morphism in $[h]$ can be interpreted as a composite of g with f.

Homotopy category hC for $C \in q\mathrm{Cat}$ has $\mathrm{obj}(hC) = C_0$ and morphism sets $\hom_{hC}(x,y) := \hom_C(x,y)/\approx$

$\to$ $\pi : C \longrightarrow N(hC)$ identity on $C_0 = N(hC)_0$, and send $f \in C_1$ to $[f]$; in particular, for an n-cell a, $\pi(a) \in N(hC)_n$ with $\pi(a)_{i-1,i} = \pi(a_{i-1,i})$.

Prop. For $C \in q\mathrm{Cat}$, hC is its fundamental category: any $\phi : C \longrightarrow ND$ factors uniquely thru π.

Proof. $\psi : N(hC) \longrightarrow ND$ uniquely determined by values on vertices and edges:

$x \in C_0$, $\psi(x) = \phi(x)$

$[f] \in hC_1$, $\psi([f]) = \phi(f)$ is well-defined;

these assignments are forced by π since it is identity on vertices and surjective on edges. $\square$

$\Rightarrow$ $h : q\mathrm{Cat} \rightleftarrows \mathrm{Cat} : N$, $h \dashv N$

Observation: Let $C' \subseteq C$ be a subcat of ∞-cats.

If $[f] = [g] \in \hom_{hC}(x, y)$, then $f \in C'_1 \Leftrightarrow g \in C'_1$

$$\begin{tikzcd} & x \arrow[dr, "g"] & \\ x \arrow[ur, "x_{00}"] \arrow[rr, "f"'] \arrow[rr, phantom, shift left=3, "a"] & & y \end{tikzcd}$$

$$\begin{aligned} g \in C'_1 &\Rightarrow a|\Lambda^2_1 \subseteq C'_1 \\ &\Rightarrow \Delta^2 \subseteq C'_1 \\ &\Rightarrow f \in C'_1 \end{aligned}$$

Prop + observation give the following bijective correspondence for $C \in q\mathrm{Cat}$,

$$\{\text{(full) subcat of } C\} \longleftrightarrow \{\text{(full) subcat of } hC\}$$

Backward: given $D' \subseteq hC$, and $f : \Delta^n \to C$ whose restriction to Λ^n_j factors thru $\tilde{D}$,

$$\begin{tikzcd} \Lambda^n_j \arrow[r] \arrow[d, tail] & \tilde{D} \arrow[r, tail] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & C \arrow[d, "\pi"] \\ \Delta^n \arrow[r, dashed, "\exists!"'] \arrow[ur, dashed, "\exists!"] \arrow[urr, "f"'] & ND' \arrow[r, tail] & N(hC) \end{tikzcd}$$

$\Rightarrow$ for $D \in \mathrm{Cat}$

$$\{\text{(full) subcat of } ND\} \longleftrightarrow \{\text{(full) subcat of } D\}.$$

..

For $C \in \infty$-Cat, $f : x \to y \in C_1$ is an equivalence / isomorphism if its image in hC is an isomorphism, i.e. if there exists $g : y \to x$ s.t. $[g] \circ [f] = [1_x]$ and $[f] \circ [g] = [1_y]$ — in this case, g is an inverse of f.

↪ Inverses of a morphism in an ∞-cat are not necessarily unique, but they are defined to be unique up to htpy.

↪ A functor $F : C \to D$ of ∞-cats sends equivalences to equivalences.

A Kan Complex is an ∞-cat K with extension property for all horns, i.e. for all $0 \leq j \leq n$, $n \geq 1$

$$\mathrm{Hom}(\Delta^n, K) \longrightarrow \mathrm{Hom}(\Lambda^n_j, K) \text{ is surjective}$$

An ∞-groupoid is an ∞-cat C s.t. hC is a groupoid, i.e. an ∞-cat in which every morphism is an equivalence.

Prop. Every Kan cpx K is an ∞-groupoid.

Proof. For any $f: x \to y$ in K_1, extension along $\Lambda^2_0 \xrightarrow{a} \Delta^2$ with $a|_{\Delta^{\{0,1\}}} = f$ gives a postinverse.

$$\begin{tikzcd} & y \arrow[dr] & \\ x \arrow[ur, "f"] \arrow[rr, "x_{00}"'] \arrow[rr, phantom, shift left=4, "\tilde{a}"] & & x \end{tikzcd}$$

Every edge in K has a postinverse.
$\Rightarrow$ K is an ∞-groupoid. $\square$

Rmk. The converse holds and is due to Joyal!

For $C \in qCat$, its <u>core</u> $C^{\simeq}$ is the subcpx consisting of cells all of whose edges are equivalences, i.e.

$$\begin{tikzcd} C^{\simeq} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "\pi"] \\ (N(hC))^{\simeq} \arrow[r, hook] & N(hC) \end{tikzcd} \quad \text{pullback in } sSet$$

$\to$ $(ND)^{\simeq} \approx N(D^{\simeq})$.

Prop. $C \in \infty\text{-Cat} \Rightarrow C^{\simeq}$ is a subcat and an ∞-groupoid.
Every subcpx of C which is an ∞-groupoid is contained in $C^{\simeq}$.

$\rightsquigarrow$ $(hC)^{\simeq} \approx h(C^{\simeq})$.

$\to$ $\pi_0(D^{\simeq}) \approx \pi_0 h(D^{\simeq}) \approx \{\text{iso classes of objs in } D\}$.

Singular complex of a space

The **topological n-simplex** is

$$\Delta^n_{top} := \{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} : \sum x_i = 1, \ x_i \geq 0 \}.$$

$\rightsquigarrow \ \Delta_{top} : \Delta \longrightarrow Top, \quad \Delta_{top}[n] = \Delta^n_{top}$

For $f : [m] \longrightarrow [n]$,

$$\Delta_{top} f : (x_0, \dots, x_m) \longmapsto (y_0, \dots, y_n). \quad y_j = \sum_{f(i)=j} x_i$$

For $T \in Top$, the **singular complex $\operatorname{Sing} T$** $\in sSet$ with

$$(\operatorname{Sing} T)_n = \operatorname{Hom}_{Top}(\Delta^n_{top}, T)$$

The **topological horn** is

$$(\Lambda^n_j)_{top} := \{ x \in \Delta^n_{top} : \exists i \in [n] \setminus j \text{ s.t. } x_i = 0 \} \subseteq \Delta^n_{top}.$$

There exists a continuous retraction $\Delta^n_{top} \longrightarrow (\Lambda^n_j)_{top}$ giving rise to, via Yoneda,

$$\operatorname{Hom}(\Delta^n, \operatorname{Sing} T) \longrightarrow \operatorname{Hom}(\Lambda^n_j, \operatorname{Sing} T)$$

which is surjective for every horn.

$\Rightarrow$ $\operatorname{Sing} T$ is a Kan complex and thus an ∞-groupoid.

$h \operatorname{Sing} T \simeq \pi_1 T$,

$\pi_0 \operatorname{Sing} T = \{ \text{path components of } T \}$

The **k-skeleton $sk_k X$** of $X \in sSet$ is the subcpx

$$(sk_k X)_n = \bigcup_{0 \le j \le k} \{ yf \mid y \in X_j , f : [n] \to [j] \}$$

$\rightsquigarrow$ $sk_{k-1} X \subseteq sk_k X$, $X = \bigcup_k sk_k X$, $sk_{n-1} \Delta^n = \partial \Delta^n$.

Prop. Skeletal filtration: for any $X \in sSet$,

$$\begin{tikzcd} \coprod_{a \in X_k^{nd}} \partial \Delta^k \arrow[r] \arrow[d] & sk_{k-1} X \arrow[d] \\ \coprod_{a \in X_k^{nd}} \Delta^k \arrow[r] & sk_k X \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Geometric realization $\|-\| : sSet \to Top$

$$\|X\| := \mathrm{Coeq} \left[\coprod_{f : [m] \to [n]} X_n \times \Delta^m_{top} \overset{(f(x), t)}{\underset{(x, f(t))}{\rightrightarrows}} \coprod_{[p]} X^p \times \Delta^p_{top} \right]$$

- $\|-\| : sSet \underset{\perp}{\rightleftarrows} Top : Sing(\ )$,
- $\|\Delta^n\| = \Delta^n_{top}$, $\|\partial \Delta^n\| = \partial \Delta^n_{top}$
- For a group G , $B(-) := \|N(-)\|$ the classifying space.

$$\Rightarrow \quad \begin{tikzcd} \coprod_{a \in X_k^{nd}} \partial \Delta^k_{top} \arrow[r] \arrow[d] & \|sk_{k-1} X\| \arrow[d] \\ \coprod_{a \in X_k^{nd}} \Delta^k_{top} \arrow[r] & \|sk_k X\| \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Thus, $\|X\| = \bigcup_k \|sk_k X\|$ is a CW-cpx whose cells are in bijection of non-deg. cells in X.

Thank you for listening!

Next time: join, slices; (co)limits in $\mathcal{Q}$Cat;

Joyal lifting.

Appendix

I messed up the domain/codomain of face/degeneracy operator in the talk. (I had the actural assignment correct.) Let me rectify my presentation;

For a simplicial set $X : \Delta^{op} \longrightarrow Set$, the <u>face operator</u>

$$d_i : [n-1] \longrightarrow [n]$$

s.t.

$$d_i(j) = \begin{cases} j & j < i \\ j+1 & j \geq i \end{cases}$$

So $d^i = \langle 0, \cdots, i-1, \hat{i}, i+1, \cdots, n \rangle$. As a simplicial operator, it acts on an n-cell $a \in X_n$ via

$$a \longmapsto a d_i \quad \in X_{n-1}.$$

Rmk: A common source of confusion is the effects of a simplicial operator on a cell from the right and on the representable objects Δ^n from the left.

This is essentially the Yoneda lemma:

$$\mathrm{Hom}_{sSet}(\Delta^m, \Delta^n) \approx \mathrm{Hom}_{\Delta}([m],[n])$$

$$a_* = (g \longmapsto g^* a = a g) \longleftarrow\!\!\shortmid \; a$$

for g any k-cell of Δ^n. any k.

When $m = n-1$, $(d_i)_* : \Delta^{n-1} \longrightarrow \Delta^n$ as you learned in a standard first course in AT:

$$(d_i)_* : \Delta^{n-1}_{top} \longrightarrow \Delta^n_{top}$$

Yoneda Lemma, 1-categorically:

For $F : C^{op} \longrightarrow Set$ in Cat and any $x \in ob C$,

$$\phi_x : Hom_{Fun(C^{op}, Set)}(Hom_C(-,x), F) \xrightarrow{\simeq} F(x)$$
$$\eta \longmapsto \eta(id_x)$$

is a bijection.

Proof: Define φ_x s.t. for any $a \in F(x)$ and $z \in C$,

$$\varphi_x(a) : Hom_C(z, x) \longrightarrow F(z), \quad g \longmapsto \underset{\overset{\parallel}{F(g)a}}{g^* a} = ag$$

is the inverse of ϕ_x. g acts on a from the right

Exe. Verify that φ_x is an inverse of ϕ_x for each x.

Cor. For any $F = Hom_C(-, y)$, we have a bijection

$$Hom_{Fun(C^{op}, Set)}(Hom_C(-, x), Hom_C(-, y)) \xrightarrow{\simeq} Hom_C(x, y)$$
$$a_* = (g \longmapsto g^* a = ag) \longleftarrow a$$

a acts on g from the left.

The previous remark is a special case where $C = \Delta$, $x = [m]$, $y = [n]$.

Rmk. Notice that I have labeled d_i, s_i in subscripts here following [kerodon.net, 000E] for the reason we have just explained. But in the talk, I did superscripts following Charles: as simplicial operators, they act on cells from the right. I consider both conventions reasonable.