iWoAT Summer School on Chromatic Homotopy Theory and Higher (Infinity-Categorical) Algebra

L4: 1-category theory of ∞-categories, II

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Yoneda Lemma, 1-categorically:

Prop. For $F : C^{op} \longrightarrow \mathrm{Set}$ in Cat and any $x \in \mathrm{ob}\, C$,

$$\phi_x : \mathrm{Hom}_{\mathrm{Fun}(C^{op},\, \mathrm{Set})}(\mathrm{Hom}_C(-,x), F) \xrightarrow{\simeq} F(x)$$

$$\eta \longmapsto \eta(id_x)$$

$$a_* = (g \longmapsto g^* a = ag) \longleftarrow\!\shortmid \; a : \psi_x$$

This prop is proven effortlessly once we have produced a pointwise inverse ψ_x of ϕ_x 1-categorically b/c natural trans between ordinary cats can be completely recovered from its values on objects:

given $\alpha : f_0 \Rightarrow f_1$ s.t. $\alpha(c) : f_0(c) \longrightarrow f_1(c)$ is iso for every object c, we get $f_0 \Leftarrow f_1 : \beta$ just by $\beta(c) = \alpha(c)^{-1}$.

There are at least two sources of difficulties when we generalize this proof to ∞-cats as we saw in L2:

① $n=1$: an inverse of edges is NOT necessarily unique

② $n>1$: one also needs to define the functor in higher dimensions!

It is remarkable that we have this "pointwise criterion for isomorphisms" in q cats:

For $C \in qCat$ and $X \in sSet$,

an edge in $Fun(X, C)$ is an iso iff it is a pointwise isomorphism/equivalence.

- what is $Fun(X, C)$ and is it a qcat?
- This result is a corollary of today.

What we also observed 1-categorically:

$P(C) = Fun(C^{op}, Set)$, the presheaf 1-cat on C.

$$Hom_{P(C)}(Hom_C(-, x), Hom_C(-, y)) \xrightarrow{\simeq} Hom_C(x, y)$$

$$a_* = (g \longmapsto g^* a = ag) \longleftarrow\!\!\shortmid \; a$$

Cor. $C \longrightarrow P(C)$ is a full faithful embedding.

$x \longmapsto Hom_C(-, x)$

∞-cat of ∞-groupoids!

L6, L9: $C \in Cat_\infty$, $P(C) = Fun(C^{op}, \mathcal{S}) \in Cat_\infty$

Hom-sets $\rightsquigarrow$ mapping spaces

$map_C(-,-) : C^{op} \times C \longrightarrow \mathcal{S}$

Yoneda Embedding holds for $C \longrightarrow P(C)$.

Given $X, Y \in sSet$, the function complex $Fun(X,Y)$ is a simplicial set with

$$Fun(X,Y)_n = Hom(\Delta^n \times X, Y)$$

and $\delta : [m] \longrightarrow [n]$ induces

$$Hom(\delta \times id_X, Y) : Hom(\Delta^n \times X, Y) \longrightarrow Hom(\Delta^m \times X, Y).$$

$\rightsquigarrow$ vertices are precisely the set of maps $X \longrightarrow Y$ in sSet.
morphisms are nat. transformations defined in L2.

$\rightsquigarrow$ $Fun : sSet^{op} \times sSet \longrightarrow sSet$ is a functor.

$\Rightarrow$ $Fun(X \times Y, Z) \xrightarrow{\approx} Fun(X, Fun(Y,Z))$

since $Hom(X \times Y, Z) \xrightarrow{=} Hom(X, Fun(Y,Z))$

$$Fun(X \times Y, Z)_n = Hom(\Delta^n \times X \times Y, Z)$$
$$\shortparallel$$
$$Fun(X, Fun(Y,Z))_n = Hom(\Delta^n \times X, Fun(Y,Z))$$

+ Yoneda lemma

$\rightarrow$ For any $C, D \in Cat$, $Fun(NC, ND) \approx N Fun(C,D)$

$$\begin{aligned} Fun(NC, ND)_n &= Hom(\Delta^n \times NC, ND) \\ &= Hom(N[n] \times NC, ND) \\ &= Hom(N([n] \times C), ND) \\ &= Hom_{Cat}([n] \times C, D) \\ &= Hom_{Cat}([n], Fun(C,D)) \\ &= N Fun(C,D)_n \end{aligned}$$

Q: When is $Fun(X,Y) \in \infty\text{-}Cat$?

Lifting properties.

For a category, e.g. sSet, which has all small colimits, a weakly saturated class $\mathcal{A}$ is a class of morphisms which contains all isos and is closed under cobase change, transfinite composition, (coproducts) & retracts.

- cobase change:

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\ B \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & B' \end{tikzcd}$$

$f \in \mathcal{A} \Rightarrow f' \in \mathcal{A}$

- transfinite composition: for any ordinal λ (well-ordered set) and functor $X : \lambda \to \mathrm{Set}$, if for every $i \in \lambda$ with $i \neq 0$,

$$\mathrm{colim}_{j<i} X(j) \longrightarrow X(i)$$

is in A, $X(0) \longrightarrow \mathrm{colim}_{j \in \lambda} X(j)$ is in A.

- retract:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, "id"] & A' \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r] \arrow[rr, bend right, "id"'] & B' \arrow[r] & B \end{tikzcd}$$

$g \in \mathcal{A} \Rightarrow f \in \mathcal{A}$.

Given a class of maps S, its weak saturation $\bar{S}$ is the smallest weakly saturated class containing S.

Given morphisms $f : A \to B$, $g : X \to Y$, a lifting problem for (f, g) is a pair of morphisms (u, v) s.t. $vf = gu$. A lift is a morphism s with $sf = u$ and $gs = v$.

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"'] & X \arrow[d, "g"] \\
B \arrow[r, "v"'] \arrow[ur, dashed, "s"] & Y
\end{tikzcd}$$

- We write $f \boxslash g$ if every lifting problem admits a lift.

$f \boxslash g \iff \operatorname{Hom}(B, X) \longrightarrow \operatorname{Hom}(A, X) \times_{\operatorname{Hom}(A,Y)} \operatorname{Hom}(B, Y)$ surjective.

- Also write $A \boxslash B$ if $a \boxslash b$ for all $a \in A$, $b \in B$.

- right complement $A^{\boxslash} = \{ g \mid a \boxslash g \text{ for all } a \in A \}$

  left complement ${}^{\boxslash}A = \{ f \mid f \boxslash a \text{ for all } a \in A \}$

↪ Prop. $A^{\boxslash}$, ${}^{\boxslash}A$ are weakly saturated.

↪ $A \subseteq B \Rightarrow A^{\boxslash} \supseteq B^{\boxslash} \quad {}^{\boxslash}A \supseteq {}^{\boxslash}B$

$\Rightarrow \quad A \subseteq {}^{\boxslash}(A^{\boxslash}), \qquad A \subseteq ({}^{\boxslash}A)^{\boxslash}$

$\Rightarrow \quad A^{\boxslash} \supseteq ({}^{\boxslash}(A^{\boxslash}))^{\boxslash}, \qquad A^{\boxslash} \subseteq ({}^{\boxslash}(A^{\boxslash}))^{\boxslash}$

$\Rightarrow \quad A^{\boxslash} = ({}^{\boxslash}(A^{\boxslash}))^{\boxslash}, \qquad {}^{\boxslash}A = {}^{\boxslash}(({}^{\boxslash}A)^{\boxslash})$

Small object argument. Let S be a set of morphisms in sSet. Every map f in sSet admits a factorization

$$f = pj, \quad j \in \overline{S}, \ p \in S^{\boxslash}.$$

$$\Rightarrow \quad \overline{S} = {}^{\boxslash}(S^{\boxslash})$$

That is, each S gives a weak factorization system $(\overline{S}, S^{\boxslash})$ for sSet.

Rmk. This also works for $\mathcal{P}$ the class of $0 \to P$ for P projective in an abelian category $\mathcal{C}$ with enough projectives, where

P projective := LLP w.r.t. epimorphisms

$\mathcal{P}^{\boxslash}$ = class of epimorphisms

$\overline{\mathcal{P}}$ = monomorphisms with projective cokernel.

$\text{InnHorn} := \{ \Lambda^n_j \subseteq \Delta^n \mid 0<j<n, \ n \geq 2 \}$

$\overline{\text{InnHorn}}$ =: class of inner anodynes

$\text{InnFib} := \text{InnHorn}^{\boxslash}$ = class of inner fibrations

$\Rightarrow$ $(\overline{\text{InnHorn}}, \text{InnFib})$ is a weak factorization system.

$\rightsquigarrow$ $C \in \text{qCat} \Leftrightarrow C \to * \in \text{InnFib}$

$\Leftrightarrow$ If $A \to B$ in $\overline{\text{InnHorn}}$, any map $A \to C$ in sSet has an extension to some $B \to C$.

⇝ Examples. • $\Lambda^n_S = \cup_{i \in S} \Delta^{[n] \setminus i}$, $S \subseteq [n]$

$\Lambda^n_S \hookrightarrow \Delta^n$ generalized horn is in $\overline{\mathrm{InnHorn}}$.

- $C' \subseteq C$ subcat is in InnFib.
- for $C \in \mathrm{qCat}$, $D \in \mathrm{Cat}$, any $C \to D \in \mathrm{InnFib}$.

$$\begin{tikzcd}
\Lambda^n_j \arrow[r] \arrow[d] & C \arrow[d] \\
\Delta^n \arrow[ur, dashed, "\exists"] \arrow[r, dashed, "\exists !"'] & ND
\end{tikzcd}$$

→ $f : X \to Y \in \overline{\mathrm{InnHorn}} \Rightarrow hf : hX \longrightarrow hY$ is an isomorphism.

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & N(hX) \arrow[d, shift left, "N(hf) \in \mathrm{InnFib}"] \\
Y \arrow[ur, dashed] \arrow[r] & N(hY) \arrow[u, dashed, shift left, "\exists !"]
\end{tikzcd}$$

also used universal property of $h(-)$.

Boundary of Δ^n: $\partial\Delta^n = \bigcup_{j \in [n]} \Delta^{[n] \setminus j}$ union of codim 1 faces

$$(\partial\Delta^n)_k = \{ f : [k] \longrightarrow [n] \mid f[k] \neq [n] \}$$

$\Rightarrow$ $\partial\Delta^0 = \phi$, $\partial\Delta^n$ is the largest proper subcpx of Δ^n (it does not contain $id_{[n]}$)

Define $\mathrm{Cell} := \{ \partial\Delta^n \subseteq \Delta^n \mid n \geq 0 \}$, $\mathrm{TrivFib} := \mathrm{Cell}^{\boxslash}$

Cell $\subseteq$ monomorphisms

where the RHS is weakly saturated.

$\Rightarrow$ $\overline{\text{Cell}} \subseteq$ monomorphisms   Converse ?

Prop. $\overline{\text{Cell}} =$ monomorphisms

Proof uses skeletal filtration.

k-skeleton $sk_k X$ of $X \in sSet$ is the subcpx

$$(sk_k X)_n = \bigcup_{0 \le j \le k} \{ f(y) \mid y \in X_j, f : [n] \to [j] \}$$

$\rightsquigarrow$ $sk_{k-1} X \subseteq sk_k X$,

$X = \bigcup_k sk_k X$,

$sk_{n-1} \Delta^n = \partial \Delta^n$.

Prop. Skeletal filtration:

for any $X \in sSet$, $A \subseteq X$ subcpx,

$$\begin{tikzcd} \coprod_{a \in X_k^{nd} \setminus A_k^{nd}} \partial \Delta^k \arrow[r] \arrow[d] & A \cup sk_{k-1} X \arrow[d] \\ \coprod_{a \in X_k^{nd} \setminus A_k^{nd}} \Delta^k \arrow[r] & A \cup sk_k X \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \qquad (*)$$

Cor. $C \to * \in \text{TrivFib} \Rightarrow C \in \text{Kan}$.

Enriched lifting.

Given $f: A \longrightarrow B$, $g: K \longrightarrow L$, $h: X \longrightarrow Y$ in sSets, the pushout-product $f \square g$ is the unique map fitting in

$$\begin{tikzcd}
A \times K \arrow[r, "g_*"] \arrow[d, "f_*"'] & A \times L \arrow[d] \arrow[ddr, bend left, "f_*"] & \\
B \times K \arrow[r] \arrow[drr, bend right, "g_*"'] & A \times L \sqcup_{A \times K} B \times K \arrow[dr, "f \square g"] & \\
& & B \times L
\end{tikzcd}$$

;

the pullback-hom $h^{\square g}$ is the unique map fitting in

$$\begin{tikzcd}
\mathrm{Fun}(L,X) \arrow[dr, "h^{\square g}"] \arrow[drr, bend left, "g^*"] \arrow[ddr, bend right, "h_*"'] & & \\
& \mathrm{Fun}(L,Y) \times_{\mathrm{Fun}(K,Y)} \mathrm{Fun}(K,X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(K,X) \arrow[d, "h_*"] \\
& \mathrm{Fun}(L,Y) \arrow[r, "g^*"] & \mathrm{Fun}(K,Y)
\end{tikzcd}$$

On vertices, $h^{\square g}$ is

$$\begin{aligned} \mathrm{Hom}(L,X) &\longrightarrow \mathrm{Hom}(L,Y) \times_{\mathrm{Hom}(K,Y)} \mathrm{Hom}(K,X) \\ s &\longmapsto (hs, sg) \end{aligned}$$

$$\begin{tikzcd}
K \arrow[r, "u"] \arrow[d, "g"'] & X \arrow[d, "h"] \\
L \arrow[r, "v"'] \arrow[ur, dashed, "s"] & Y
\end{tikzcd}$$

Thus $h^{\square g}$ is surjective on vertices iff $g \boxslash h$. So it is an "enriched" lifting problem of $g \boxslash h$.

Adjunction of lifting problems

Prop. $(f \square g) \boxslash h \Leftrightarrow f \boxslash (h^{\square g})$.

$$\begin{tikzcd}
A\times L \coprod_{A\times K} B\times K \arrow[r, "{(u,v)}"] \arrow[d, "f\square g"'] & X \arrow[d, "h"] & A \arrow[r, "\tilde{u}"] \arrow[d, "f"'] & \mathrm{Fun}(L,X) \arrow[d, "h^{\square g}"] \\
B\times L \arrow[r, "w"'] \arrow[ur, dashed, "s"] & Y & B \arrow[r, "{(\tilde{w},\tilde{v})}"'] \arrow[ur, dashed, "\tilde{s}"] & \mathrm{Fun}(L,Y) \times_{\mathrm{Fun}(K,Y)} \mathrm{Fun}(K,X)
\end{tikzcd}$$

In particular, when $K=\emptyset$, $Y=*$, $f\square g = f\times L$

$$(A\times L \xrightarrow{f\times L} B\times L) \boxslash (X \longrightarrow *)$$

$$\Leftrightarrow (A \xrightarrow{f} B) \boxslash (\mathrm{Fun}(L,X) \longrightarrow *)$$

⇝ Let $\mathcal{C} := \mathrm{Fun}([1], s\mathrm{Set})$ be the arrow category of sSet.

Then $(\square, \emptyset \subseteq \Delta^0)$ defines a symm. monoidal structure on $\mathcal{C}$ and $(-)\square g \dashv (-)^{\square g}$

Prop. For any set V of morphisms in sSet, $V\mathrm{Fib} := V^{\boxslash}$

Then $S \square T \subseteq \overline{U}$ implies $\overline{S} \square \overline{T} \subseteq \overline{U} = {}^{\boxslash}U\mathrm{Fib}$,

Cor. $\overline{S} \boxslash U\mathrm{Fib}^{\square \overline{T}} \Rightarrow U\mathrm{Fib}^{\square\overline{T}} \subseteq S\mathrm{Fib}$

$\overline{T} \boxslash U\mathrm{Fib}^{\square \overline{S}} \Rightarrow U\mathrm{Fib}^{\square\overline{S}} \subseteq S\mathrm{Fib}$

Prop. $\overline{\mathrm{InnHorn}} \square \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}$; $\overline{\mathrm{Cell}} \square \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Cell}}$

Cor. $\mathrm{InnFib}^{\square\overline{\mathrm{Cell}}} \subseteq \mathrm{InnFib}$

$\mathrm{InnFib}^{\square\overline{\mathrm{InnHorn}}} \subseteq \mathrm{TrivFib}$; $\mathrm{TrivFib}^{\square\overline{\mathrm{Cell}}} \subseteq \mathrm{TrivFib}$.

Thm. For $C \in q\mathrm{Cat}$, $L \in s\mathrm{Set}$, $\mathrm{Fun}(L,C) \in q\mathrm{Cat}$.

Proof.

$$\begin{array}{ccc} \overline{\mathrm{InnHorn}} \,\square\, \overline{\mathrm{Cell}} & & \mathrm{InnFib} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (\Lambda^n_j \xrightarrow{f} \Delta^n) \,\square\, (\emptyset \longrightarrow L) & \boxslash & (C \longrightarrow *) \end{array}$$

$$\iff (\Lambda^n_j \xrightarrow{f} \Delta^n) \boxslash (\mathrm{Fun}(L,C) \longrightarrow *) .$$

□

Restriction along $j : A \longrightarrow B$ an inner anodyne or monomorphism induces trivial fibration or inner fibration of functor qcats:

$$j^* = (p : C \longrightarrow *)^{\square j} : \mathrm{Fun}(B,C) \longrightarrow \mathrm{Fun}(A,C) .$$

Another characterization of a qcat.

We saw $\overline{\mathrm{InnHorn}} \,\square\, \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}$; in fact,

$$\overline{\{\Lambda^2_1 \subseteq \Delta^2\} \,\square\, \mathrm{Cell}} = \overline{\mathrm{InnHorn}} \text{ due to Joyal.}$$

Cor. $C \in s\mathrm{Set}$ is an ∞-cat.

$$\iff f = (C \longrightarrow *)^{\square\{\Lambda^2_1 \subseteq \Delta^2\}} : \mathrm{Fun}(\Delta^2, C) \longrightarrow \mathrm{Fun}(\Lambda^2_1, C)$$

is a trivial fibration.

Proof. For all $n \geq 0$

$$(\partial\Delta^n \longrightarrow \Delta^n) \,\square\, (\Lambda^2_1 \longrightarrow \Delta^2) \boxslash (C \longrightarrow *)$$

$$\iff (\partial\Delta^n \longrightarrow \Delta^n) \boxslash f$$

Thus

$$(C \longrightarrow *) \in \mathrm{InnFib}$$

$$\iff (C \longrightarrow *) \in (\{\Lambda^2_1 \subseteq \Delta^2\} \,\square\, \mathrm{Cell})^{\boxslash}$$

$$\iff f \in \mathrm{TrivFib} = \mathrm{Cell}^{\boxslash} .$$

□

Composition revisited.

We saw in talk 1 that composition is unique up to homotopy:

$$[g]\circ[f] = [gf] \text{ is well-defined.}$$

Now we can do better.

- Every trivial fibration admits a section.

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & X \arrow[d, "h"] \\
Y \arrow[r, "id"'] \arrow[ur, dashed, "s"] & Y
\end{tikzcd}$$

- $\overset{\text{spine}}{I^2} = \Lambda^2_1 \longrightarrow \Delta^2 \in \mathrm{InnHorn}$ induces trivial fibration r_* in functor cats for which we pick a section s fitting in

$$\begin{tikzcd}
\mathrm{Fun}(\Lambda^2_1, C) \arrow[r, shift left, "s"] & \mathrm{Fun}(\Delta^2, C) \arrow[l, shift left, "r_*"] \arrow[r, "r'"] & \mathrm{Fun}(\Delta^{\{0,2\}}, C)
\end{tikzcd}$$

$\Rightarrow$ sr' is a composition functor.

- n-fold composition functor: $\overset{\text{spine}}{I^n} \longrightarrow \Delta^n \in \mathrm{InnHorn}$

  gives rise to

$$\begin{tikzcd}
\mathrm{Fun}([1],C) \times_C \mathrm{Fun}([1],C) \times_C \cdots \times_C \mathrm{Fun}([1],C) \arrow[d, equal] & \\
\mathrm{Fun}(I^n, C) \arrow[d, shift left, "s"] & \\
\mathrm{Fun}(\Delta^n, C) \arrow[u, shift left, "r_*"] \arrow[r, "r'"] & \mathrm{Fun}(\Delta^{\{0,n\}}, C)
\end{tikzcd}$$

Rmk: This definition is not unique since s is not. But we will see all such functors are "naturally isomorphic."

Categorical Equivalences

"the correct notion of equivalences for qcats"

- A categorical inverse to $f : C \longrightarrow D$ of qcats is a $g : D \longrightarrow C$ s.t. gf is naturally isomorphic to 1_C and fg is naturally isomorphic to 1_D.
- A map $f : X \longrightarrow Y$ in sSet is a categorical equivalence if for every ∞-cat E,

$$f^* : \mathrm{Fun}(Y, E) \longrightarrow \mathrm{Fun}(X, E) \text{ of qcats}$$

admits a categorical inverse.

Prop. For $f : C \longrightarrow D$ of qcats, f admits a categorical inverse iff it is a categorical equivalence.

Cor. For $f : X \longrightarrow Y$ in sSet, TFAE:

1) f is a categorical equivalence;
2) for any qcat C,

$$hf^* : h\mathrm{Fun}(Y, C) \longrightarrow h\mathrm{Fun}(X, C)$$

is an equivalence of 1-cats.

3) for any qcat C,

$$\pi_0 \mathrm{Fun}(Y, C)^{\simeq} \longrightarrow \pi_0 \mathrm{Fun}(X, C)^{\simeq}$$

is a bijection.

Rmk. Prop lets you reduce 3) $\Rightarrow$ 1) to f between qcats.

3) is the definition Lurie uses for cat equiv in kerodon.

Prop. TrivFib $\subseteq$ CatEq. Cor. $\overline{\text{InnHorn}} \subseteq$ CatEq.

Fix $p : X \longrightarrow S \in$ TrivFib.

① If $S = *$, p is a categorical equivalence exhibiting X as a contractible (π_0 singleton) Kan cpx.

② Recall for any $Y \in$ sSet,

$$p_* = p^{\square(\emptyset \subset Y)} : \mathrm{Fun}(Y, X) \longrightarrow \mathrm{Fun}(Y, S)$$

is also a trivial fibration. (Same holds for InnFib.)

③ In the two pullbacks in sSets, vertical maps on the left

$$\begin{tikzcd}
\text{parametrizes sections of } p \arrow[r, squiggly] & \mathrm{Fun}_{/S}(S, X) \arrow[r] \arrow[d, "\mathrm{CatEq} \ni"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{Fun}(S, X) \arrow[d, "p_*"', "\in \mathrm{TrivFib}"] \\
 & \{\mathrm{id}\} \arrow[r] & \mathrm{Fun}(S, S)
\end{tikzcd}$$

$$\begin{tikzcd}
sp \simeq \mathrm{id}_X \text{ in} \arrow[r, squiggly] & \mathrm{Fun}_{/S}(X, X) \arrow[r] \arrow[d, "\mathrm{CatEq} \ni"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{Fun}(X, X) \arrow[d, "p_*"', "\in \mathrm{TrivFib}"] \\
 & \{p\} \arrow[r] & \mathrm{Fun}(X, S)
\end{tikzcd}$$

are contractible ∞-groupoids by ①.

$\Rightarrow p : X \longrightarrow S \in$ CatEq. $\square$

Rmk: If p is a trivial fib between ∞cats, any section is its categorical inverse and any two sections are nat. isomorphic, living in a contractible Kan cpx.

$\Rightarrow$ Composition is unique up to a contractible space of choices.

Examples of cat. equiv.

F = free monoid on one generator g.

obj set is $\{g\}$, morphism set is $\{g^n \mid n \geq 0\}$.

$\Rightarrow (NF)_d = \{(g^{m_1}, \dots, g^{m_d}) \mid m_i \geq 0\}$

The map $\gamma : S^1 = \Delta^1/\partial\Delta^1 \longrightarrow NF$ is in $\overline{\text{InnHorn}}$.

simplicial circle $\quad \overline{\langle 01 \rangle} \longmapsto g$

(proof is an explicit computation using skeletal filtration.)

$\rightsquigarrow$ NF is the free monoid generated as a qcat by S^1.

$\Rightarrow$ Enriched Lifting + $N(-)$ monoidal gives

$\gamma^{\times n} : (S^1)^{\times n} \longrightarrow N(F^{\times n})$ is in $\overline{\text{InnHorn}} \subseteq \text{Cat}_{Eq}$.

$\rightsquigarrow$ free comm monoid generated as a qcat.

Join

For $A, B \in \mathrm{Cat}$, the join $A \star B$ is the category

$$\mathrm{ob}(A \star B) = \mathrm{ob}A \sqcup \mathrm{ob}B$$

$$\mathrm{mor}(A \star B) = \mathrm{mor}A \sqcup (\mathrm{ob}A \times \mathrm{ob}B) \sqcup \mathrm{mor}B$$

with

$$\mathrm{Hom}_{A \star B}(x,y) = \begin{cases} \mathrm{Hom}_{A \star B}(x,y) & x, y \in \mathrm{ob}A \\ \mathrm{Hom}_{A \star B}(x,y) & x, y \in \mathrm{ob}B \\ \{*\} & x \in \mathrm{ob}A \quad y \in \mathrm{ob}B \\ \emptyset & x \in \mathrm{ob}B \quad y \in \mathrm{ob}A \end{cases}$$

left cone : $A^{\triangleleft} = [0] \star A$ freely adjoins an initial obj.

right cone : $A^{\triangleright} = A \star [0]$ freely adjoins a terminal obj.

Ordered disjoint union

$$\sqcup : \Delta \times \Delta \longrightarrow \Delta, \qquad [p] \sqcup [q] = [p+1+q]$$

$$(0, 1, \cdots, p, 0, \cdots, q)$$

We extend Δ to $\Delta_+ = \Delta^{\triangleleft}$ by adjoining $[-1] := \emptyset$

Thus $(\sqcup, [-1])$ is a monoidal structure on Δ_+.

For each map $f : [p] \longrightarrow [q_1] \sqcup [q_2]$ in Δ_+, there's a unique decomposition $[p] = [p_1] \sqcup [p_2]$ s.t. f (uniquely) decomposes as

$$f = f_1 \sqcup f_2, \quad f_i : [p_i] \longrightarrow [q_i].$$

For $X, Y \in sSet$, **join** $X \star Y$ is the simplicial set with

$$(X \star Y)_n := \coprod_{[n] = [n_1] \sqcup [n_2]} X_{n_1} \times Y_{n_2}$$

over $[n_i] \in \mathrm{ob}\, \Delta_+$, and we set $X_{-1} = * = Y_{-1}$

For $(x, y) \in X_{n_1} \times Y_{n_2} \subseteq (X \star Y)_n$ and $f : [m] \longrightarrow [n]$,

$$(x, y) f = (x f_1, y f_2),$$

where $f = f_1 \sqcup f_2$, $f_i : [m_i] \longrightarrow [n_i]$ is the unique decomposition of f over $[n] = [n_1] \sqcup [n_2]$.

e.g. $(X \star Y)_0 = X_0 \sqcup Y_0$

$(X \star Y)_1 = X_1 \sqcup (X_0 \times Y_0) \sqcup Y_1$

$(X \star Y)_2 = X_2 \sqcup (X_1 \times Y_0) \sqcup (X_0 \times Y_1) \sqcup Y_2$

$\rightsquigarrow$ $X \longrightarrow X \star Y \longleftarrow Y$ subcpx.

$\hookrightarrow$ $(\star, \emptyset =: \Delta^{-1})$ is a monoidal structure on sSet and $X \star -$, $- \star X$ preserve pushout.

$\hookrightarrow$ $\Delta^p \star \Delta^q = \Delta^{p+1+q}$

left cone : $A^{\triangleleft} = \Delta^0 \star A$ **right cone** : $A^{\triangleright} = A \star \Delta^0$

$\rightsquigarrow$ $N(A \star B) = N(A) \star N(B)$

for $B = [0]$,

$\Rightarrow$ $N(A^{\triangleleft}) = (NA)^{\triangleleft}$, $N(A^{\triangleright}) = (NA)^{\triangleright}$

Example : $(\partial\Delta^n)^\triangleleft = \Delta^0 \star \partial\Delta^n \approx \Lambda^{n+1}_0$

$(\partial\Delta^n)^\triangleright = \partial\Delta^n \star \Delta^0 \approx \Lambda^{n+1}_{n+1}$

$$\begin{tikzcd} 0 \arrow[dr] & \\ & v \\ 1 \arrow[ur] & \end{tikzcd}$$

$(\partial\Delta^1)^\triangleright = \Lambda^2_2$

$$\begin{tikzcd} & 0 \arrow[dr] & \\ v \arrow[ur] \arrow[rr] \arrow[dr] & & 2 \\ & 1 \arrow[ur] & \end{tikzcd}$$

$(\Lambda^2_2)^\triangleleft = \Delta^1\times\Delta^1 = \Delta^2 \cup_{\Delta^1} \Delta^2$

Lemma. Maps $f : K \longrightarrow X \star Y$ of sSets are in bijection with the tuples

$(\pi : K \longrightarrow \Delta^1,\ f_{\{0\}} : K^{\{0\}} \longrightarrow X,\ f_{\{1\}} : K^{\{1\}} \longrightarrow Y)$

where $K^{\{j\}} = \pi^{-1}(\{j\})$.

Prop. Join of qcats is a qcat.

Proof. Use the lemma and $\Lambda^n_j \longrightarrow \Delta^1$ is either const at 0, or const at 1, or nonconst.

$\Rightarrow$ $\Lambda^n_j \longrightarrow X \star Y$ factors as

$\Lambda^n_j \longrightarrow X \star \Delta^{-1} \longrightarrow X \star Y$

or $\Lambda^n_j \longrightarrow \Delta^{-1} \star Y \longrightarrow X \star Y$

or $\Lambda^n_j \longrightarrow \Delta^k \star \Delta^{n-1-k} \longrightarrow X \star Y$.

$\square$

Slices

coslice

For $p : S \longrightarrow X$, $X_{p/}$ is the slice-under simplicial set with

$$(X_{p/})_n = \mathrm{Hom}_{sSet_{S/}}(S \star \Delta^n, X)$$

slice

For $q : T \longrightarrow X$, $X_{/q}$ is the slice-over simplicial set with

$$(X_{/q})_n = \mathrm{Hom}_{sSet_{T/}}(\Delta^n \star T, X)$$

Notices that $X \star -$, $- \star X$ do not preserve colimits b/c $X \star \emptyset = X \neq \emptyset$, $\emptyset \star X = X \neq \emptyset$. But

Prop. For every $X \in sSet$,

$$X \star - \ , \ - \star X : sSet \longrightarrow sSet_{X/}$$

preserve colimits and they have right adjoints

$$(p : S \longrightarrow X) \longmapsto X_{p/} : sSet_{S/} \longrightarrow sSet$$

$$(q : T \longrightarrow X) \longmapsto X_{/q} : sSet_{T/} \longrightarrow sSet$$

In general, we have bijective correspondences, known as the "join/slice adjunction"

$$\left\{ \begin{tikzcd} S \star \emptyset = S \arrow[r, "p"] \arrow[d] & X \\ S \star K \arrow[ur, dashed] & \end{tikzcd} \right\} \Longleftrightarrow \{ K \dashrightarrow X_{p/} \}$$

$$\left\{ \begin{tikzcd} \emptyset \star T = T \arrow[r, "q"] \arrow[d] & X \\ K \star T \arrow[ur, dashed] & \end{tikzcd} \right\} \Longleftrightarrow \{ K \dashrightarrow X_{/q} \}$$

- Nerve preserves slices: for $p : A \to C$ in Cat,

$$N(C_{p/}) \simeq NC_{Np/}, \quad N(C_{/p}) \simeq NC_{/Np}$$

- Functoriality of slice : given $T \xrightarrow{j} S \xrightarrow{p} X \xrightarrow{f} Y$

$\rightsquigarrow$

$$\begin{tikzcd}
X_{/p} \arrow[r] \arrow[d] & Y_{/fp} \arrow[d] \\
X_{/pj} \arrow[r] & Y_{/fpj}
\end{tikzcd}$$

commutative square.

$$\begin{tikzcd}
T \arrow[r, "j"] \arrow[d, hook] & S \arrow[r, "p"] \arrow[d, hook] & X \arrow[r, "f"] & Y \\
K \star T \arrow[r, "K \star j"] & K \star S \arrow[ur, dashed, "\tilde{u}"'] & &
\end{tikzcd}$$

slice-under

$\rightsquigarrow$

$T \star K$, $S \star K$

$u : K \to X_{/f}$ restricting to $K \xrightarrow{u} X_{/f} \longrightarrow X_{/pfj}$

corresponds to $\tilde{u}$ restricting to $p\tilde{u}(K \star j)$.

Letting $K = \Delta^n$ in the slice-over ssets, we get assignments on n-cells:

- $\emptyset \xrightarrow{j} S \xrightarrow{p} X \xrightarrow{\overset{\mathrm{id}_X}{\overset{\parallel}{f}}} X$

  $\rightsquigarrow$ $X_{/p} \longrightarrow X$

  $\tilde{x} : \Delta^n \star S \to X \longmapsto \tilde{x}|\Delta^0 \star \emptyset$

- $\emptyset \xrightarrow{j} S \xrightarrow{p} X \xrightarrow{f} Y$

  $\rightsquigarrow$ $X_{/f} \longrightarrow Y_{/fp}$

  $\tilde{x} : \Delta^n \star S \to X \longmapsto f\tilde{x} : \Delta^n \star S \to Y$

Slice categories of a qcat is a qcat.

Lifting properties of join/slices

- Given $i : A \longrightarrow B$, $j : K \longrightarrow L$, the pushout-join is

$$i \boxed{\star} j : (A \star L) \underset{A \star K}{\sqcup} (B \star K) \longrightarrow B \star L$$

induced by $(i \star L, B \star j)$.

- Note. $\boxed{\star}$ is not symm since $\star$ is not.

- For $0 \leq j \leq n$,

$$(\Lambda^n_j \subset \Delta^n) \boxed{\star} (\emptyset \subset \Delta^0) \approx (\Lambda^{n+1}_j \subset \Delta^{n+1})$$

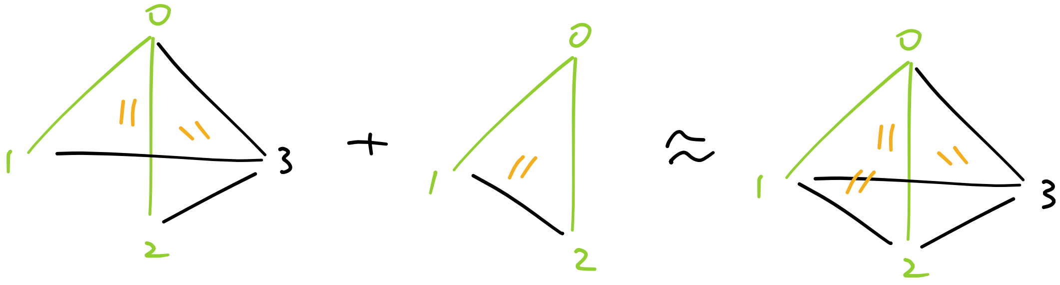
$$(\emptyset \subset \Delta^0) \boxed{\star} (\Lambda^n_j \subset \Delta^n) \approx (\Lambda^{1+n}_{1+j} \subset \Delta^{1+n})$$

$$(\partial\Delta^n \subset \Delta^n) \boxed{\star} (\emptyset \subset \Delta^0) \approx (\partial\Delta^{n+1} \subset \Delta^{n+1})$$

$$(\emptyset \subset \Delta^0) \boxed{\star} (\partial\Delta^n \subset \Delta^n) \approx (\partial\Delta^{1+n} \subset \Delta^{1+n})$$

e.g. $(\Lambda^2_0 \subset \Delta^2) \boxed{\star} (\emptyset \subset \Delta^0)$ is the map

$$\Lambda^2_0 \star \Delta^0 \underset{\Lambda^2_1 \cdot \star \emptyset}{\sqcup} \Delta^2 \star \emptyset \approx \Lambda^3_0 \longrightarrow \Delta^3$$



For general k,

$$(\Lambda^n_j \subset \Delta^n) \boxed{\star} (\partial\Delta^k \subset \Delta^k) \approx (\Lambda^{n+1+k}_j \subset \Delta^{n+1+k})$$

$$(\partial\Delta^k \subset \Delta^k) \boxed{\star} (\Lambda^n_j \subset \Delta^n) \approx (\Lambda^{k+1+n}_{k+1+j} \subset \Delta^{k+1+n})$$

$$(\partial\Delta^n \subset \Delta^n) \boxed{\star} (\partial\Delta^k \subset \Delta^k) \approx (\partial\Delta^{n+1+k} \subset \Delta^{n+1+k})$$

given $T \xrightarrow{j} S \xrightarrow{p} X \xrightarrow{f} Y$,

$$\begin{tikzcd} X_{p/} \arrow[r] \arrow[d] & Y_{fp/} \arrow[d] \\ X_{pj/} \arrow[r] & Y_{fpj/} \end{tikzcd}$$

commutes.

the pullback slices are

$f^{\boxast_p j} : X_{p/} \longrightarrow X_{pj/} \times_{Y_{fpj/}} Y_{fp/}$, $f^{j \boxast_p} : X_{/p} \longrightarrow X_{/pj} \times_{Y_{/fpj}} Y_{/fp}$.

Prop. $(i \boxast j) \boxslash f \iff i \boxslash (f^{\boxast_q j})$ for all $q : L \longrightarrow X$

$\iff j \boxslash (f^{i \boxast_p})$ for all $p : B \longrightarrow X$

Proof for first $\iff$:

$$\begin{tikzcd} \emptyset \star L \arrow[r] \arrow[rr, bend left, "q"] & A \star L \sqcup_{A \star K} B \star K \arrow[r, "{(u,v)}"] \arrow[d, "i \boxast j"'] & X \arrow[d, "f"] \\ & B \star L \arrow[r, "w"'] \arrow[ur, dashed, "s"] & Y \end{tikzcd}$$

$$\iff \begin{tikzcd} A \arrow[r, "\tilde{u}"] \arrow[d, "i"'] & X_{/q} \arrow[d, "f^{\boxast_q j}"] \\ B \arrow[r, "{(\tilde{v}, \tilde{w})}"'] \arrow[ur, dashed, "\tilde{s}"] & X_{/qj} \times_{Y_{/fqj}} Y_{/fq} \end{tikzcd} .$$

□

Prop. $\overline{\mathrm{RHorn}} \boxast \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}$; $\overline{\mathrm{Cell}} \boxast \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Cell}}$

$\overline{\mathrm{Cell}} \boxast \overline{\mathrm{LHorn}} \subseteq \overline{\mathrm{InnHorn}}$;

Cor. Join preserves monomorphisms.

For $i : A \longrightarrow B$ in $\overline{\mathrm{Cell}}$, factor $S \star i$ as

$$S \star A \longrightarrow S \star A \cup_{\emptyset \star A} \emptyset \star B \xrightarrow{(\emptyset \subseteq S) \star i} S \star B$$

cobase change of a mono is mono. □

Cor. Slice preserves trivial fibrations.

$$X_{p/} \xrightarrow{f^{\boxast}_{p}(\emptyset \subseteq S)} X \times_{Y} Y_{fp/} \longrightarrow Y_{fp/}$$

base change of trivial fib

for $X \xrightarrow{f} Y$ trivial fib. □

Cor. Given $T \xrightarrow{j} S \xrightarrow{p} C$ with $C \in \mathrm{qCat}$ (so $f : C \longrightarrow *$ is an inner fibration.) We have pullback slices

$$\ell = f^{\boxast_p j} : C_{p/} \longrightarrow C_{pj/} \ , \quad r = f^{j \boxast_p} : C_{/p} \longrightarrow C_{/pj}$$

1) $j \in \overline{\mathrm{Cell}} \Rightarrow \ell \in \mathrm{LFib}$, $r \in \mathrm{RFib}$

1') $T = \emptyset$, $(\ell : C_{p/} \longrightarrow C) \in \mathrm{LFib} \subseteq \mathrm{TrivFib}$

$(r : C_{/p} \longrightarrow C) \in \mathrm{RFib} \subseteq \mathrm{TrivFib}$

$$\begin{tikzcd}
\Lambda^n_j \arrow[r] \arrow[d] & C_{p/} \arrow[d, "\in \mathrm{TrivFib}"] \\
\Delta^n \arrow[r, dashed, "\exists"'] \arrow[ur, dashed, "\exists"] & C
\end{tikzcd}$$

$\Rightarrow$ Slices of a qcat is a qcat!

Colimits & Limits

An **initial object** of a qcat C is an $x \in C_0$ s.t. every $f : \partial\Delta^n \longrightarrow C$ $(n \geq 1)$ with $f|\{0\} = x$ admits an extension to $f' : \Delta^n \longrightarrow C$.

A **terminal object** is an initial object in C^{op}.

$$\begin{tikzcd}
\{0\} \arrow[r, hook] \arrow[rr, bend left, "x"] & \partial\Delta^n \arrow[r] \arrow[d, hook] & C \\
& \Delta^n \arrow[ur, dashed, "\exists"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\{n\} \arrow[r, hook] \arrow[rr, bend left, "y"] & \partial\Delta^n \arrow[r] \arrow[d, hook] & C \\
& \Delta^n \arrow[ur, dashed, "\exists"'] &
\end{tikzcd}$$

Given $f : K \longrightarrow C$ for C a qCat, a **colimit** of f is an initial object of the slice qcat $C_{f/}$. That is, a colimit $\hat{f} : K^{\triangleright} \longrightarrow C$ extends f and lift exists in every diagram (for $n \geq 1$) of the form

$$\begin{tikzcd}
K \star \{0\} \arrow[r, hook] \arrow[rr, bend left, "\hat{f}"] & K \star \partial\Delta^n \arrow[r] \arrow[d, hook] & C \\
& K \star \Delta^n \arrow[ur, dashed, "\exists"'] &
\end{tikzcd}$$

$\hat{f}$ is also referred to as a **colimit cone of f**, with **cone point** $\hat{f}|\emptyset \star \{0\}$.

Similarly, a **limit** of f is a terminal object of $C_{/f}$.

Example. ① For $K=\emptyset$, $f:\emptyset \longrightarrow C$ for C qcat.

$\Rightarrow$ $C_{/f} = C$ and a colimit of f is precisely an initial object of C.

② $K = \Lambda^2_2$; $K^{\triangleleft} \approx \Delta^1 \times \Delta^1 \longrightarrow C$ is a pullback diagram in C.

$$\begin{tikzcd}
 & 0 \arrow[dr] & \\
v \arrow[ur] \arrow[rr] \arrow[dr] & & 2 \\
 & 1 \arrow[ur] &
\end{tikzcd}$$

Prop. $x \in C_0$ is initial $\Leftrightarrow$ $(C_{x/} \longrightarrow C) \in \mathrm{TrivFib}$

$x \in C_0$ is terminal $\Leftrightarrow$ $(C_{/x} \longrightarrow C) \in \mathrm{TrivFib}$.

Proof. For all $n \geq 0$,

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "f"] \arrow[d, hook] & C_{x/} \arrow[d, "\pi"] \\
\Delta^n \arrow[r, "g"'] \arrow[ur, dashed] & C
\end{tikzcd}
\Leftrightarrow
\begin{tikzcd}
\Delta^0 * \emptyset \arrow[r] \arrow[rr, bend left, "x"] & \emptyset * \Delta^n \sqcup_{\emptyset * \partial\Delta^n} \Delta^0 * \partial\Delta^n \arrow[r, "(\tilde{f}, g)"] \arrow[d, "i \boxtimes j"'] & C \\
 & \Delta^0 * \Delta^n \arrow[ur, dashed] &
\end{tikzcd}$$

where $(i : \emptyset \subset \Delta^0) \boxtimes (j : \partial\Delta^n \subset \Delta^n) \approx (\partial\Delta^{1+n} \subset \Delta^{1+n})$.

$$\Leftrightarrow
\begin{tikzcd}
\{0\} \arrow[r, hook] \arrow[rr, bend left, "x"] & \partial\Delta^{1+n} \arrow[r] \arrow[d, hook] & C \\
 & \Delta^{1+n} \arrow[ur, dashed, "\exists"'] &
\end{tikzcd}$$

$\square$

Cor. Let $\tilde{f} : K^{\triangleright} \longrightarrow C$ for C a qCat and write $f := \tilde{f}|K$.

Then $\tilde{f}$ is a colimit iff $(C_{\tilde{f}/} \longrightarrow C_{f/}) \in \mathrm{TrivFib}$;

$\tilde{f} : K^{\triangleleft} \longrightarrow C$ is a limit iff $(C_{/\tilde{f}} \longrightarrow C_{/f}) \in \mathrm{TrivFib}$.

Proof. $C_{\tilde{f}/} = (C_{f/})_{\tilde{f}/}$, where $\tilde{f} \in (C_{f/})_0$. $\square$

Prop. For $C \in q\mathrm{Cat}$, let C^{init} and C^{term} denote respectively the full subcat spanned by initial objs & terminal objs. Then each of C^{init} and C^{term} are either empty or categorically equivalent to Δ^0. That is, initial and terminal objs are unique to unique isomorphism.

Proof.
- $C^{\mathrm{term}} = ((C^{op})^{\mathrm{init}})^{op}$.
- For $n \geq 1$, $(\partial\Delta^n)_0 = (\Delta^n)_0 \neq \emptyset$.

For any $\partial\Delta^n \xrightarrow{f} C^{\mathrm{init}}$,

$$\begin{tikzcd}
\{0\} \arrow[r, hook] \arrow[rr, bend left, "f|\{0\}"] & \partial\Delta^n \arrow[r, "f"] \arrow[d, hook] & C^{\mathrm{init}} \arrow[r] & C \\
 & \Delta^n \arrow[ru, dashed, "\exists"] \arrow[rru, dashed, "\exists"'] & &
\end{tikzcd}$$

If $C^{\mathrm{init}} \neq \emptyset$, extn exists also for $n=0$.

Thus $(C^{\mathrm{init}} \longrightarrow *) \in \mathrm{TrivFib}$, so C^{init} is a contractible Kan cpx. □

Cor. colimits and limits are unique.

Rmk. Need Joyal lifting for some other properties of initial & terminal objs.

Joyal extention & lifting theorem !!!

Joyal extention theorem. Let $C \in q\mathrm{Cat}$ and $f \in C_1$, TFAE.

1) The edge represented by f is an isomorphism.
2) Every $a : \Lambda^n_0 \longrightarrow C$ with $n \geq 2$ s.t. $f = a|\Delta^{\{0,1\}}$ admits an extension to a map $\Delta^n \longrightarrow C$
3) Every $b : \Lambda^n_n \longrightarrow C$ with $n \geq 2$ s.t. $f = b|\Delta^{\{n-1,n\}}$ admits an extension to a map $\Delta^n \longrightarrow C$

special case of :

Joyal lifting theorem. Let $p : C \longrightarrow D$ be an inner fibration between qCats and $f \in C_1$ with $p(f)$ being an isomorphism in D. TFAE.

1) The edge f is an isomorphism in C.
2) For all $n \geq 2$, lift exists in any

$$\begin{tikzcd}
\Delta^{\{0,1\}} \arrow[r, hook] \arrow[rr, bend left, "f"] & \Lambda^n_0 \arrow[r] \arrow[d, hook] & C \arrow[d, "p"] \\
& \Delta^n \arrow[r] \arrow[ur, dashed, "\exists"] & D
\end{tikzcd} .$$

3) For all $n \geq 2$, lift exists in any

$$\begin{tikzcd}
\Delta^{\{n-1,n\}} \arrow[r, hook] \arrow[rr, bend left, "f"] & \Lambda^n_n \arrow[r] \arrow[d, hook] & C \arrow[d, "p"] \\
& \Delta^n \arrow[r] \arrow[ur, dashed, "\exists"] & D
\end{tikzcd} .$$

Corollaries of Joyal's theorems!

Exe. Every simplicial set has extension property against 1-dim'l horns $\Lambda^1_j \longrightarrow \Delta^1$.

Theorem. An ∞-groupoid is a Kan cpx.

Prop. If $f : x \to y$ edge in $C \in$ qcat is an isomorphism,

$$\begin{tikzcd} C_{/x} \arrow[r, leftarrow] \arrow[r, dashed, bend left] & C_{/f} \arrow[r] & C_{/y} \end{tikzcd}$$

form a zigzag of cat equiv and so $C_{/x}$ and $C_{/y}$ are categorically equivalent.

Note: The composition functor can be thought of realizing

$$\begin{tikzcd} (c \xrightarrow{g} x) \arrow[r, mapsto] & (c \xrightarrow{gf} y) \\ C_{/x} \arrow[u, phantom, sloped, "\in"] & C_{/y} \arrow[u, phantom, sloped, "\in"] \end{tikzcd}$$

Thm. Invariance of initial/terminal obj.

For $f \in C_1$ for C a qcat $\leftrightsquigarrow$ $\tilde{f} \in (C_{x/})_0$,

$\tilde{f}$ initial in $C_{x/}$ $\Leftrightarrow$ f is an isomorphism.

Thm. "pointwise criterion for isomorphisms" in qcats as advertised in the intro:

For $C \in$ qCat and $X \in$ sSet, an edge in $\mathrm{Fun}(X, C)$ is an iso iff it is a pointwise equivalence.

Thank you for listening !