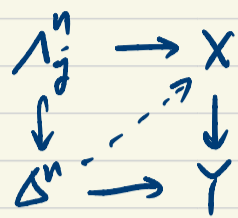


1. left/right fibration, (co)Cartesian fibration

recall left horn: $\Lambda_j^n \quad 0 \leq j < n$ right horn: $\Lambda_j^n \quad 0 < j \leq n$.

a map of ssets $X \rightarrow Y$ is a left fibration if it satisfies the right lifting property with respect to left horn inclusions.



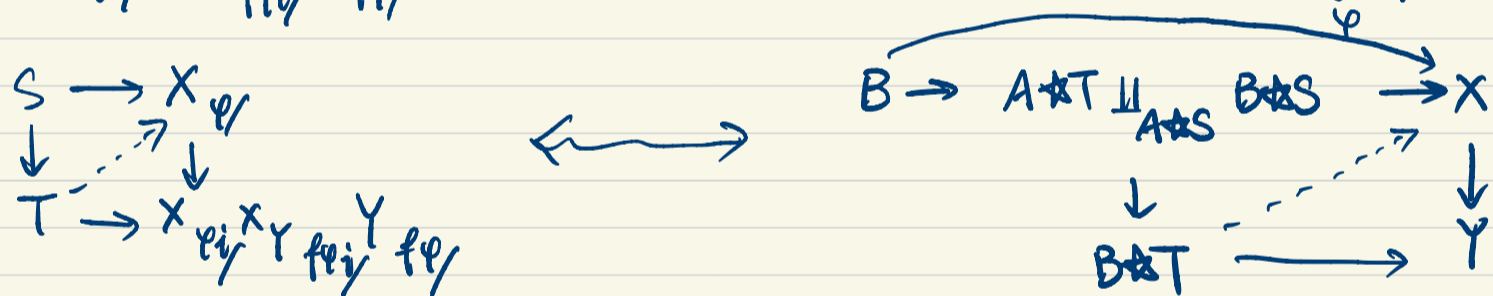
we will give defn for coCartesian fibration then come back to make connections / motivations.

also recall. Let S be a sset. we have adjunctions.

$$S \star - : \mathbf{sSet} \rightleftarrows \mathbf{sSet}_S : \begin{array}{c} S \xrightarrow{p} X \\ \downarrow \\ X_p \end{array} \leftarrow \text{sset given by } n \mapsto \text{Hom}_{\mathbf{sSet}_S}(S \star \Delta^n, X)$$

$$- \star S : \mathbf{sSet} \rightleftarrows \mathbf{sSet}_S : \begin{array}{c} S \xrightarrow{p} X \\ \downarrow \\ X_p \end{array}$$

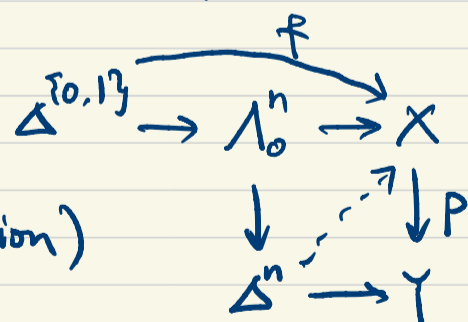
So here is a lemma: For $A \xrightarrow{i} B \xrightarrow{p} X \xrightarrow{f} Y$ in \mathbf{sSet} , \exists an induced map $X_{p/} \rightarrow X_{p/} \times_{X_Y} Y_{f/}$. And there is a bijection of lifting problems.



we give 2 defn for p-(co)Cartesian morphisms.

defn 1: $\Delta \xrightarrow{f} X$ is called a p-coCartesian morphism if for $n \geq 2$, any lifting problem $\Delta^{[0,1]} \rightarrow \Lambda_0^n \rightarrow X$ $X \rightarrow Y$ is an inner fibration.

(easier to see the connection)



admits a solution.

defn 2: (in HTT) let $X \rightarrow Y$ be an inner fibration and let $f: \alpha \rightarrow \gamma$ be a morphism in X . Then f is p-coCartesian iff the functor

$$X_{p/} \rightarrow X_{\alpha/} \times_{Y_{f(\alpha)/}} Y_{f/}$$

is a trivial fibration.

check: Since trivial fibration, consider the lifting problem

$$\begin{array}{ccc}
 S & \Delta^n & \rightarrow X_p \\
 \downarrow & \nearrow & \downarrow \\
 T & \Delta^n & \rightarrow X_{p/q} \times_{Y_{p/q}} Y_{p/q}
 \end{array}$$

$$\begin{array}{ccccc}
 A & \xrightarrow{f} & B & \xrightarrow{p} & X & \xrightarrow{f} & Y \\
 \Delta^0 & \xrightarrow{g} & \Delta^1 & \xrightarrow{f} & X & \xrightarrow{p} & Y
 \end{array}$$

$$\begin{array}{ccc}
 \Delta^1 * \Delta^n \perp \Delta^0 * \Delta^n & \xrightarrow{\Delta^0 * \Delta^n} & X \\
 \downarrow & \nearrow & \downarrow \\
 \Delta^1 * \Delta^n & \xrightarrow{\quad} & Y
 \end{array}$$

the restriction to Δ^1 is given by f .

this map is iso to $\Lambda_0^{n+2} \rightarrow \Delta^{n+2}$ and inclusion $\Delta^1 \rightarrow \Lambda_0^{n+2}$ is $\Delta^{[0,1]}$.

a map of sSet $X \xrightarrow{p} Y$ is a coCartesian fibration if it is inner fibration and every lifting problem

$$\begin{array}{ccc}
 \{0\} & \rightarrow & X \\
 \downarrow & \nearrow & \downarrow \\
 \Delta^1 & \rightarrow & Y
 \end{array}$$

admits a p -coCartesian solution in X .

left fibration is coCartesian fibration because if $X \rightarrow Y$ left fib, every mor in X is p -coCartesian.

2. motivation

from covering space? we would like fiber over a point be nice.

Assume $E \xrightarrow{p} C$ in ω -Cat, think about the square

$$\begin{array}{ccc}
 E_{\pi} & \rightarrow & E \\
 \downarrow & \lrcorner & \downarrow p \\
 * & \rightarrow & C
 \end{array}$$

if E_x is nice. for instance, it is also an ω -cat.

then this assignment gives a functor $C \rightarrow \text{Cat}_\omega$.

fact: if $E \xrightarrow{p} C$ inner fibration, then E_x is an ω -Cat for any $x \in C$

if further, $E \xrightarrow{p} C$ left/right fibration, then E_x is an ω -Gpd. (Kan ops).

So this construction gives connection between

? fibrations and functors from C to $\text{Cat}_\omega / \text{Gpd}_\omega$

Can we say more about "the collection of E_{π} "?

fiber over Δ^1 : from point to edge.

$$\begin{array}{ccc}
 ? & \rightarrow & E \\
 \downarrow & \lrcorner & \downarrow p \\
 \Delta^1 & \rightarrow & C
 \end{array}$$

Fact: if $E \rightarrow C$ is a Cartesian fibration, then for every morphism $x \rightarrow y$ in C , we obtain a functor $E_x \rightarrow E_y$

(the composition is well defined up to contractible).

3. Straightening - Unstraightening equivalence.

What to expect? Given an ∞ -cat C , we would like to look at Cat_∞ / C .

let $\text{coCart}(C) :=$ subcategory of Cat_∞ / C , obj: $E \rightarrow C$. mor: morphisms of

coCartesian fibrations $\begin{array}{ccc} X & \xrightarrow{f} & X' \\ P \downarrow & & \downarrow P' \end{array}$ f sends p -coCartesian morphisms to p' -coCartesian morphisms.

let $\text{LFib}(C) :=$ full subcategory of $\text{coCart}(C)$ on left fibrations.

$\text{LFib}(C) \subseteq \text{Cat}_\infty / C$ is also a full subcategory.

$\text{Spc} :=$ ∞ -category of spaces. (simplicial category with obj: CW-cplx.
 $\text{Gpd}_\infty = \text{Kan}$. \leftarrow take coherent nerve. mapping sset := Sing of mapping space.

Thm (Lurie). \exists equivalences of ∞ -categories.

$$\text{coCart}(C) \simeq \text{Fun}(C, \text{Cat}_\infty) \quad \text{which restricts to}$$

$$\text{an equivalence: } \text{LFib}(C) \simeq \text{Fun}(C, \text{Spc})$$

And for X an ∞ -gpd, it restricts to $\text{Spc}/X \simeq \text{Fun}(X, \text{Spc})$.

idea: construct a functor $C \rightarrow \text{Cat}_\infty$ from each coCartesian fibration

This gives the equivalence on objects.

informal construction: $E \rightarrow C$ coCartesian fibration. we have.

- for each $x \in C$, we have an ∞ -category E_x .

- for each $x \xrightarrow{f} y \in C$, an object z in E_x , can choose a p -coCartesian

left. $z \rightarrow w$ of f . denote $w = f_!(z)$.

$$\begin{array}{ccc} z & \xrightarrow{f_!} & w & \rightarrow & E \\ & & \downarrow & & \downarrow P \\ x & \xrightarrow{f} & y & \rightarrow & C \end{array}$$

- for another $z' \in E_x$, and an edge $z \xrightarrow{\alpha} z'$, we can choose another

p -coCartesian left $z' \rightarrow w' = f_!(z')$ with

$$\begin{array}{ccc} z & \xrightarrow{\alpha} & z' \\ \downarrow & & \downarrow \\ f_!(z) & \dashrightarrow & f_!(z') \\ & & f_!(x) \end{array}$$

the choice of dotted arrow is contractible.

so for edge $x \xrightarrow{f} y$, gives functor $\mathcal{E}_x \xrightarrow{f!} \mathcal{E}_y$.

sketch proof of $f!$ is a functor: $\text{Fun}_f^{\text{cc}}(\Delta', \mathcal{E}) = \text{map to } p\text{-cocartesian mor in } \mathcal{E} \text{ under } f$.

$\text{Fun}_f^{\text{cc}}(\Delta', \mathcal{E}) \rightarrow \mathcal{E}_x$ by taking the source is a trivial fibration.

choosing a section, gives the composite

$$f! : \mathcal{E}_x \rightarrow \text{Fun}_f^{\text{cc}}(\Delta', \mathcal{E}) \xrightarrow{\text{target}} \mathcal{E}_y.$$

\downarrow this map is adjoint to a map

$$\mathcal{E}_x \times \Delta' \rightarrow \mathcal{E}. \text{ s.t. restricts to } \mathcal{E}_x \times \{1\} \xrightarrow{f!} \mathcal{E}$$

and $\mathcal{E}_x \times \Delta' \rightarrow \mathcal{E}$

$$\begin{array}{ccc} \downarrow & & \downarrow p \\ \Delta' & \xrightarrow{f} & \mathcal{C}. \end{array} \text{ commute.}$$

and for each $z \in \mathcal{E}_x$, the resulting morphism $\Delta' \rightarrow \mathcal{E}$ is p -cocartesian $z \rightarrow f!(z)$

left to show; the association $f \mapsto f!$ is functorial in f .

then move to general construction.

Ex) consider $\mathcal{C} = \text{Cat}_{\infty}$ and the identity functor.

the corresponding cocartesian fibration is called the universal cocartesian fibration.

This is a functor $(\text{Cat}_{\infty})_{*/1} \rightarrow \text{Cat}_{\infty}$.

where the ∞ -Cat $(\text{Cat}_{\infty})_{*/1}$ has obj: pair (C, α) $\alpha \in C$

$$\text{mor: } \begin{array}{c} (C, \alpha) \\ \downarrow (F, \alpha) \\ (D, \beta) \end{array} \text{ where } F: C \rightarrow D \\ \alpha: F\alpha \rightarrow \beta \text{ in } D.$$

4. stack and Hopf algebroids.

a stack is a sheaf of groupoids that satisfy **effective descent** is a glueing condition

Ex). let $X \in \text{Top}$. To each open $U \subseteq X$, we assign the groupoid $\text{Bund}_n(U)$

with obj: real n -plane bundles over U

mor: bundle iso over U

As U varies, Bund_n is a sheaf of groupoids.

effective descent: Suppose $\{V_i\}$ is an open cover of U , bundles ξ_i over V_i

and $\phi_{ij}: \xi_i|_{V_i \cap V_j} \xrightarrow{\cong} \xi_j|_{V_i \cap V_j}$ over $V_i \cap V_j$.

satisfy the cocycle condition.

Warning: the assignment $U \mapsto \text{Bund}_n(U)$ is not a functor.

$$\begin{array}{ccc} & \longleftarrow \mathcal{E} & \\ \text{if} & & (gf)^*\mathcal{E} \cong f^*g^*\mathcal{E} \text{ not equal.} \\ & U_1 \xrightarrow{f} U_2 \xrightarrow{g} U_3 & \end{array}$$

So if replace Gpd with $\infty\text{-Gpd}$, there's no problem.

stacks form a full subcategory of the category of presheaves of (∞) -groupoids.

↑
there is a model str here where
stacks = fibrant.

let R a comm ring and $\text{Spec}(R) :=$ representable functor $\text{Hom}(R, -)$ on the cat of comm rings.

A **Hopf algebroid** over a comm ring R is a groupoid obj. in the category of (graded) comm R -algebras.

That is a pair (A, Γ) of comm R -algebras with structure maps s.t. for any other $B \in \text{CAlg}_R$, the sets $\text{Hom}(A, B)$ and $\text{Hom}(\Gamma, B)$ are the obj's and mors of a groupoid. Γ is a left and right A -mod.

structure maps:

$$\begin{array}{ccccc}
 & & \Gamma & \xleftarrow{c \cdot \Gamma} & P \otimes_R \Gamma & \xrightarrow{\Gamma \cdot c} & \Gamma & & \\
 & & \uparrow & \swarrow & \downarrow & \searrow & \uparrow & & \\
 & & \mathcal{M}_R & & P \otimes_A \Gamma & & \mathcal{M}_L & & \\
 & & & & \uparrow \Delta & & & & \\
 A & \xleftarrow{\varepsilon} & P & \xrightarrow{\varepsilon} & A & & & &
 \end{array}$$

let $G_P(R)$ be the groupoid has $\text{Hom}(A, B)$ as obj. and $\text{Hom}(\Gamma, B)$ as mor. $\text{Spec}(R) \mapsto G_P(R)$ gives a (pre)sheaf.

And if $R \rightarrow S$ faithfully flat extension. then

$$G_P(R) \rightarrow G_P(S) \rightrightarrows G_P(S \otimes_R S) \quad \text{is an equalizer diag. of gpds.}$$

but G_P is almost never a stack. stackify G_P , get $\mathcal{M}(A, \Gamma)$ for the stack

Ex). degenerate case. $A = \Gamma = R$. then $\mathcal{M}(A, \Gamma) = \text{Spec}(R)$. is a stack.

given $R \rightarrow S$ a faithfully flat extension, $A \twoheadrightarrow S$ and $\phi: \Gamma \rightarrow S \otimes_R S$ satisfies the cocycle condition and $A \xrightarrow[\mathcal{M}_R]{\mathcal{M}_L} P \xrightarrow{\phi} S \otimes_R S$ are identity.

then $\mathcal{M}(A, \Gamma)$ is a moduli/classifying obj. for elements of G_P .

connection to formal group law: (bedrock example).

let L the Lazard ring. $\text{Hom}(L, R)$ is naturally iso to the set of f.g.l.s. over R . $L \cong MU^*$ (Quillen's thm, Guchuan).

$$W = L[b_0^{\pm 1}, b_1, b_2, \dots] \quad \text{note that, if } b_0 = 1, \text{ it is } MU^* MU$$

Write \mathcal{M}_{fgc} for the resulting stack. get from the pair (L, W) .

morphisms $\text{Spec}(R) \rightarrow \mathcal{M}_{\text{fgc}}$ classify equivalence classes of f.g.-ks over faithfully flat extensions of R