1. left / right fibration, (co)Cartesian fibration

recall left horn: Λ^n_j $\quad 0 \le j < n$ $\qquad$ right horn: Λ^n_j $\quad 0 < j \le n$.

a map of sSets $X \to Y$ is a left fibration if it satisfies the right lifting property with respect to left horn inclusions.

$$\begin{tikzcd} \Lambda^n_j \arrow[r] \arrow[d, hook] & X \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

we will give defn for coCartesian fibration then come back to make connections / motivations.

also recall. Let S be a sSet. we have adjunctions.

$$S \star - : sSet \rightleftarrows sSet_{S/} : \begin{array}{c} S \xrightarrow{p} X \\ \downarrow \\ X_{p/} \end{array}$$

$X_{p/}$ ← sSet given by $n \mapsto Hom_{sSet_{S/}}(S \star \Delta^n, X)$

$$- \star S : sSet \rightleftarrows sSet_{S/} : \begin{array}{c} S \xrightarrow{p} X \\ \downarrow \\ X_{/p} \end{array}$$

So here is a lemma: For $A \xrightarrow{i} B \xrightarrow{\varphi} X \xrightarrow{f} Y$ in sSet, $\exists$ an induced map $X_{\varphi/} \to X_{\varphi i/} \times_{Y_{f\varphi i/}} Y_{f\varphi/}$. And there is a bijection of lifting problems.

$$\begin{tikzcd} S \arrow[r] \arrow[d] & X_{\varphi/} \arrow[d] \\ T \arrow[r] \arrow[ur, dashed] & X_{\varphi i/} \times_{Y_{f\varphi i/}} Y_{f\varphi/} \end{tikzcd} \quad \longleftrightarrow \quad \begin{tikzcd} B \arrow[r] \arrow[rr, bend left, "\varphi"] & A \star T \amalg_{A \star S} B \star S \arrow[r] \arrow[d] & X \arrow[d] \\ & B \star T \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

we give 2 defn for p-(co)Cartesian morphisms.

defn 1: $\Delta^1 \xrightarrow{f} X$ is called a p-coCartesian morphism if for $n \ge 2$, any lifting problem

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[r] \arrow[rr, bend left, "f"] & \Lambda^n_0 \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ & \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

$X \xrightarrow{p} Y$ is an inner fibration.

(easier to see the connection)

admits a solution.

defn 2: (in HTT) let $X \xrightarrow{p} Y$ be an inner fibration and let $f: x \to y$ be a morphism in X. Then f is p-coCartesian iff the functor

$$X_{f/} \to X_{x/} \times_{Y_{p(x)/}} Y_{pf/}$$

is a trivial fibration.

check: since trivial fibration, consider the lifting problem

$$\begin{tikzcd} S \quad \partial\Delta^n \arrow[r] \arrow[d] & X_{f/} \arrow[d] \\ T \quad \Delta^n \arrow[r] \arrow[ur, dashed] & X_{x/} \times_{Y_{p(x)/}} Y_{pf/} \end{tikzcd}$$

$A \xrightarrow{i} B \xrightarrow{\varphi} X \xrightarrow{f} Y$

$\Delta^0 \xrightarrow{x} \Delta^1 \xrightarrow{f} X \xrightarrow{p} Y$

$\updownarrow$

$$\begin{tikzcd} \Delta^1 \star \partial\Delta^n \amalg_{\Delta^0 \star \partial\Delta^n} \Delta^0 \star \Delta^n \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^1 \star \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

the restriction to Δ^1 is given by f.

this map is iso to $\Lambda_0^{n+2} \to \Delta^{n+2}$ and inclution $\Delta^1 \to \Lambda_0^{n+2}$ is $\Delta^{\{0,1\}}$.

a map of sSet $X \xrightarrow{p} Y$ is a coCartesian fibration if it is inner fibration and every lifting problem

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^1 \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

admits a p-coCartesian solution in X.

left fibration is coCartesion fibration because if $X \to Y$ left fib, every mor in X is p-coCartesian.

2. motivation

from covering space? we would like fiber over a point be nice.

Assume $\mathcal{E} \xrightarrow{p} \mathcal{C}$ in ∞-Cat, think about the square

$$\begin{tikzcd} \mathcal{E}_x \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E} \arrow[d, "p"] \\ * \arrow[r, "x"'] & \mathcal{C} \end{tikzcd}$$

if $\mathcal{E}_x$ is nice. for instance, it is also an ∞-cat.

then this assignment gives a functor $\mathcal{C} \to \mathrm{Cat}_\infty$.

fact: if $\mathcal{E} \xrightarrow{p} \mathcal{C}$ inner fibration, then $\mathcal{E}_x$ is an ∞-Cat for any $x \in \mathcal{C}$

if further, $\mathcal{E} \xrightarrow{p} \mathcal{C}$ left/right fibration, then $\mathcal{E}_x$ is an ∞-Gpd. (Kan cplx).

So this construction gives connection between

? fibrations and functors from $\mathcal{C}$ to $\mathrm{Cat}_\infty / \mathrm{Gpd}_\infty$

Can we say more about "the collection of $\mathcal{E}_x$"?

fiber over Δ^1: from point to edge.

$$\begin{tikzcd} ? \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E} \arrow[d, "p"] \\ \Delta^1 \arrow[r] & \mathcal{C} \end{tikzcd}$$

Fact: if $\mathcal{E} \xrightarrow{p} C$ is a coCartesian fibration, then for every morphism $x \to y$ in C, we obtain a functor $\mathcal{E}_x \to \mathcal{E}_y$

(the composition is well defined up to contractible).

3. straightening-unstraightening equivalence.

what to expect? Given an ∞-cat C, we would like to look at Cat_∞/C.

let $coCart(C) :=$ subcategory of Cat_∞/C, obj: $\mathcal{E} \to C$. mor: morphisms of coCartesian fibrations

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & X' \arrow[dl, "p'"] \\ & Y & \end{tikzcd}$$

f sends p-coCartesian morphisms to p'-coCartesian morphisms.

let $LFib(C) :=$ full subcategory of $coCart(C)$ on left fibrations.

$LFib(C) \subseteq Cat_\infty/C$ is also a full subcategory.

$Spc := \infty$-category of spaces. $\begin{cases} \text{simplicial category with obj: CW-cplx.} \\ \text{mapping sSet} := \text{Sing of mapping space.} \end{cases}$

$\overset{\shortparallel}{Gpd_\infty} = Kan.$ take coherent nerve.

Thm (Lurie). $\exists$ equivalences of ∞-categories.

$coCart(C) \simeq Fun(C, Cat_\infty)$ which restricts to

an equivalence: $LFib(C) \simeq Fun(C, Spc)$

And for X an ∞-gpd, it restricts to $Spc/X \simeq Fun(X, Spc)$.

idea: construct a functor $C \to Cat_\infty$. from each coCartesian fibration

This gives the equivalence on objects.

informal construction: $\mathcal{E} \xrightarrow{p} C$ coCartesian fibration. we have.

- for each $x \in C$, we have an ∞-Category $\mathcal{E}_x$.
- for each $x \xrightarrow{f} y \in C$, an object z in $\mathcal{E}_x$, can choose a p-coCartesian

lift. $z \to w$ of f. denote $w = f_!(z)$.

$$\begin{tikzcd} z \arrow[r, "f_!"] \arrow[d, mapsto] & w \arrow[r] & \mathcal{E} \arrow[d, "p"] \\ x \arrow[r, "f"] & y \arrow[r] & C \end{tikzcd}$$

- for another $z' \in \mathcal{E}_x$, and an edge $z \xrightarrow{\alpha} z'$, we can choose another

p-coCartesian lift $z' \to w' = f_!(z')$ with

$$\begin{tikzcd} z \arrow[r, "\alpha"] \arrow[d] & z' \arrow[d] \\ f_!(z) \arrow[r, dashed, "f_!(\alpha)"'] & f_!(z') \end{tikzcd}$$

the choise of dotted arrow is contractible.

so for edge $x \xrightarrow{f} y$, gives functor $\mathcal{E}_x \xrightarrow{f_!} \mathcal{E}_y$.

sketch proof of $f_!$ is a functor: $\mathrm{Fun}^{cc}_f(\Delta^1, \mathcal{E})$ = map to p-coCartesian mor in $\mathcal{E}$ under f.

$\mathrm{Fun}^{cc}_f(\Delta^1, \mathcal{E}) \to \mathcal{E}_x$ by taking the source is a trivial fibration.

choosing a section, gives the composite

$f_! : \mathcal{E}_x \to \mathrm{Fun}^{cc}_f(\Delta^1, \mathcal{E}) \xrightarrow{\text{target}} \mathcal{E}_y$.

this map is adjoint to a map

$\mathcal{E}_x \times \Delta^1 \to \mathcal{E}$. s.t. restricts to $\mathcal{E}_x \times \{1\} \xrightarrow{f_!} \mathcal{E}$

and
$$\begin{array}{ccc} \mathcal{E}_x \times \Delta^1 & \to & \mathcal{E} \\ \downarrow & & \downarrow p \\ \Delta^1 & \xrightarrow{f} & C \end{array}$$
commute.

and for each $z \in \mathcal{E}_x$, the resulting morphism $\Delta^1 \to \mathcal{E}$ is p-coCartesian $z \to f_!(z)$

left to show; the association $f \mapsto f_!$ is functorial in f.

then move to general construction.

Ex) consider $C = \mathrm{Cat}_\infty$ and the identity functor.

the corresponding coCartesian fibration is called the universal coCartesian fibration.

This is a functor $(\mathrm{Cat}_\infty)_{*//} \to \mathrm{Cat}_\infty$.

where the ∞-Cat $(\mathrm{Cat}_\infty)_{*//}$ has obj: pair (C, x) $x \in C$

mor: $(C, x) \xrightarrow{(F, \alpha)} (\mathcal{D}, y)$ where $F : C \to \mathcal{D}$, $\alpha : Fx \to y$ in $\mathcal{D}$.

4. stack and Hopf algebroids.

a stack is a sheaf of groupoids that satisfy effective descent — is a glueing condition

Ex). let $X \in \mathrm{Top}$. To each open $U \subseteq X$, we assign the groupoid $\mathrm{Bund}_n(U)$

with obj: real n-plane bundles over U

mor: bundle iso over U

As U varies, Bund_n is a sheaf of groupoids.

effective descent: Suppose $\{V_i\}$ is an open cover of U, bundles ξ_i over V_i

and $\phi_{ij} : \xi_i|_{V_i \cap V_j} \xrightarrow{\simeq} \xi_j|_{V_i \cap V_j}$ over $V_i \cap V_j$.

satisfy the cocycle condition.

Warning: the assignment $U \mapsto \mathrm{Bund}_n(U)$ is not a functor.

if $U_1 \xrightarrow{f} U_2 \xrightarrow{g} U_3$, ξ $\quad (gf)^*\xi \cong f^*g^*\xi$. not equal.

so if replace Gpd with ∞-Gpd, there's no problem.

stacks form a full subcategory of the category of presheaves of (∞)-groupoids.

there is a model str here where stacks = fibrant.

let R a comm ring and $\mathrm{Spec}(R) :=$ representable functor $\mathrm{Hom}(R, -)$ on the cat of comm rings.

A Hopf algebroid over a comm ring R is a groupoid obj. in the category of (graded) comm R-algebras.

That is a pair (A, Γ) of comm R-algebras. with structure maps s.t. for any other $B \in CAlg_R$, the sets $\mathrm{Hom}(A, B)$ and $\mathrm{Hom}(\Gamma, B)$ are the obj's and mors of a groupoid. Γ is a left and right A-mod.

structure maps:

$$\begin{tikzcd}
\Gamma & \Gamma \otimes_R \Gamma \arrow[l, "c \cdot \Gamma"'] \arrow[r, "\Gamma \cdot c"] \arrow[d] & \Gamma \\
& \Gamma \otimes_A \Gamma \arrow[ul, dashed] \arrow[ur, dashed] & \\
A \arrow[uu, "\eta_R"] & \Gamma \arrow[l, "\varepsilon"'] \arrow[r, "\varepsilon"] \arrow[u, "\Delta"] & A \arrow[uu, "\eta_L"']
\end{tikzcd}$$

let $G_\Gamma(R)$ be the groupoid has $\mathrm{Hom}(A, B)$ as obj. and $\mathrm{Hom}(\Gamma, B)$ as mor.

$\mathrm{Spec}(R) \mapsto G_\Gamma(R)$ gives a (pre)sheaf.

And if $R \to S$ faithfully flat extension. then

$$G_\Gamma(R) \to G_\Gamma(S) \rightrightarrows G_\Gamma(S \otimes_R S)$$ is an equilizer diag of gpds.

but G_Γ is almost never a stack. stackificate G_Γ, get $\mathcal{M}(A, \Gamma)$ for the stack

Ex). degenerate case. $A = \Gamma = R$. then $\mathcal{M}(A, \Gamma) = \mathrm{Spec}(R)$. is a stack.

given $R \to S$ a faithfully flat extension, $A \not\cong S$ and $\phi : \Gamma \to S \otimes_R S$ satisfies the cocycle condition and $A \underset{\eta_R}{\overset{\eta_L}{\rightrightarrows}} \Gamma \xrightarrow{\phi} S \otimes_R S$ are identity.

then $\mathcal{M}(A, \Gamma)$ is a moduli / classifying obj. for elements of G_Γ.

connection to formal group law: (bedrock example).

let L the Lazard ring. $\mathrm{Hom}(L, R)$ is naturally iso to the set of f.g.l.s. over R.

$L \cong MU^*$ (~~Quillen's thm~~, Guchuan).

$W = L[b_0^{\pm 1}, b_1, b_2, \dots]$ note that, if $b_0 = 1$, it is $MU^* MU$

Write $\mathcal{M}_{fgl}$ for the resulting stack. get from the pair (L, W).

morphisms $\mathrm{Spec}(R) \to \mathcal{M}_{fgl}$ classify equivalence classes of f.g.ls over faithfully flat extensions of R