∞-categories lecture 4.

adjoint functors, presentable infinity categories

Reference.

Jacob Lurie, Higher Topos Theory

Markus Land, Introduction to ∞-categories.

Ch 5. adjunction & adjoint functor thm.

Review.

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}. \qquad F \dashv G.$$

$$\mathrm{Hom}_{\mathcal{D}}(Fx, y) \cong \mathrm{Hom}_{\mathcal{C}}(x, Gy), \quad \forall\, \begin{matrix} x \in \mathcal{C} \\ y \in \mathcal{D} \end{matrix}.$$

Def. An adjunction $p: \mathcal{E} \to \Delta^1$, that is a bicartesian fibration.

Using the straightening/unstraightening techniques

$$\mathcal{E}_0 \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{E}_1$$

fibre over 0 fiber over 1

f is left adjoint to g

g is right adjoint to f

$f \dashv g$

Rmk. $f: \mathcal{C} \to \mathcal{D}$.

We say f has a right adjoint if $\exists\ p: \mathcal{E} \to \Delta^1$

$$\mathcal{C} \simeq \mathcal{E}_0 \xrightarrow{f_p} \mathcal{E}_1 \simeq \mathcal{D}.$$

$\underline{\text{Expect}}$. $\mathcal{E}_0 \underset{g}{\overset{f}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{E}_1$. $x \in \mathcal{E}_0$, $y \in \mathcal{E}_1$

$$\mathrm{map}_{\mathcal{E}_1}(fx, y) \simeq \mathrm{map}_{\mathcal{E}_0}(x, gy).$$

$(p : \mathcal{E} \to \Delta^1)$

$\underline{\text{Prop}}$. There is a natural equivalence of functors

$$\mathrm{map}_{\mathcal{E}_0}(-, g(-)) \simeq \mathrm{map}_{\mathcal{E}_1}(f(-), -).$$

$$\mathcal{E}_0^{op} \times \mathcal{E}_1 \longrightarrow \mathcal{S}$$

equivalent to $\mathcal{E}_0^{op} \times \mathcal{E}_1 \longrightarrow \mathcal{E}^{op} \times \mathcal{E} \xrightarrow{\text{mappings space}} \mathcal{S}$

construct two natural transformations

$$\tau_g : i_0 \circ g \longrightarrow i_1 \qquad \tau_f : i_0 \longrightarrow i_1 \circ f$$

$(i_0 : \mathcal{E}_0 \to \mathcal{E}, \quad i_1 : \mathcal{E}_1 \to \mathcal{E})$.

$$\begin{tikzcd} \mathcal{E}_0 \times \{0\} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ \mathcal{E}_0 \times \Delta^1 \arrow[r] \arrow[ur, dashed, "\tau_f"] & \Delta^1 \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E}_1 \times \{1\} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ \mathcal{E}_1 \times \Delta^1 \arrow[r] \arrow[ur, dashed, "\tau_g"] & \Delta^1 \end{tikzcd}$$

$$\mathcal{E} \to \Delta^1 \rightsquigarrow \mathcal{E}^{\mathcal{E}_1} \to (\Delta^1)^{\mathcal{E}_1}, \ \mathcal{E}^{\mathcal{E}_0} \to (\Delta^1)^{\mathcal{E}_0}$$

Look at

$$\mathcal{E}_0^{op} \times \mathcal{E}_1 \times \Delta^1 \xrightarrow{\mathcal{E}_0^{op} \times \tau_g} \mathcal{E}_0^{op} \times \mathcal{E} \to \mathcal{E}^{op} \times \mathcal{E} \to \mathcal{S}$$

This is a pointwise equivalence $\overset{\text{[Mar] 2.2.2.}}{\Longrightarrow}$ equivalence.

unit & counit for adjunctions.

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D} \qquad \mathcal{D}(Fx, y) \cong \mathcal{C}(x, Gy).$$

take $y = Fx$.

$x \to GFx$.

$\rightsquigarrow$ natural transformation from $Id_{\mathcal{C}} \overset{\eta = \text{unit}}{\Longrightarrow} GF$

(also, $FG \overset{\varepsilon = \text{counit}}{\Longrightarrow} Id_{\mathcal{D}}$).

$\tau_g : i_0 \circ g \to i_1$

$\tau_f : i_0 \to i_1 \circ f$

Form the following diagram:

$$\begin{tikzcd}
\mathcal{E}_1 \times \Delta^1 \arrow[r, "\tau_g"] \arrow[d, "g \times id"'] & \mathcal{E} \\
\mathcal{E}_0 \times \Delta^1 \arrow[ur, "\tau_f"'] &
\end{tikzcd}$$

e.g. restrict to $\mathcal{E}_1 \times \{0\}$.

Combine the two ~~com~~ $(\tau_g, \tau_f \circ (g \times id))$.

$$\begin{tikzcd}
\mathcal{E}_1 \times \Lambda^2_0 \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\
\mathcal{E}_1 \times \Delta^2 \arrow[r] \arrow[ur, dashed] \arrow[dr] & \Delta^1 \\
& \Delta^2 \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
& 0 \arrow[dl, "\tau_g"'] \arrow[dr, "\tau_f \circ (g \times id)"] & \\
1 & & 2
\end{tikzcd}$$

$0 \mapsto 0$

$1, 2 \mapsto 1$.

look at things pointwise

$z \in \mathcal{E}_1$

$$\begin{tikzcd}
& g(z) \arrow[dl] \arrow[dr] & \\
z \arrow[rr, dashed] & & f(g(z))
\end{tikzcd}$$

$\mathcal{E}_1$

Unit of the adjunction

Dually $\rightsquigarrow$ counit of adjunction.

unit: $\mathcal{E}_0 \times \Delta^1 \to \mathcal{E}_0$

counit. $\mathcal{E}_1 \times \Delta^1 \to \mathcal{E}_1$

[Mar]

Q: $f: \mathcal{C} \to \mathcal{D}$.

f will have a right adjoint, if for each object x in $\mathcal{D}$, you can find an object $gx \in \mathcal{C}$. as well as a universal $f(g(x)) \xrightarrow{\varepsilon_x} x$ in $\mathcal{D}$ s.t.

$$\mathrm{map}_{\mathcal{C}}(z, gx) \xrightarrow{f} \mathrm{map}_{\mathcal{D}}(f(z), f(g(x))) \xrightarrow{\varepsilon_x} \mathrm{map}_{\mathcal{D}}(f(z), x)$$

then $\exists$ right adjoint $g: \mathcal{D} \to \mathcal{C}$.

$$x \mapsto gx$$

counit of adjunction is going to be the given ε_x's.

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right adjoint exists $\Rightarrow$ unique up to equivalence

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adjunctions pass to functor categories

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "f"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left, "g"] \end{tikzcd}$$

$\eta: \mathrm{id} \to gf$.

$\Delta^1 \to \mathrm{Fun}(\mathcal{C}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{C}^K, \mathcal{C}^K)$

if K is a simplicial set $\mathcal{E}$ is an ∞-cat.

$$\begin{tikzcd} \mathrm{Fun}(K, \mathcal{C}) \arrow[r, shift left, "f_*"] \arrow[r, phantom, "\perp"] & \mathrm{Fun}(K, \mathcal{D}) \arrow[l, shift left, "g_*"] \end{tikzcd}$$

checked by exponential rule & triangle identities

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{D}, \mathcal{E}) \arrow[r, shift left, "f^*"] \arrow[r, phantom, "\perp"] & \mathrm{Fun}(\mathcal{C}, \mathcal{E}) \arrow[l, shift left, "g^*"] \end{tikzcd}$$

relation between limits & colimits.

Prop. If $\mathcal{C}$ is an ∞-category which has K-indexed colimits (K simplicial set),

$$\rightsquigarrow \quad \mathrm{colim}_K : \mathrm{Fun}(K, \mathcal{C}) \to \mathcal{C}$$
$$\perp$$
$$\mathrm{const} \quad \mathcal{C} \longrightarrow \mathrm{Fun}(K, \mathcal{C}).$$

Dual statement holds for limits.

Prop. left adjoints will preserve colimits
right adjoint will preserve limits.

Prop. const functor. preserves limits and colimits
if $\mathcal{C}$ is complete, cocomplete,

Combine w/ AFT:

$\mathcal{C}$ presentable $\quad$ const: $\mathcal{C} \to \mathrm{Fun}(K', \mathcal{C})$.
preserves colim

$\rightsquigarrow$ const has a right adjoint

$$\boxed{\text{presentable } \infty\text{-categories are complete}}$$

$\mathcal{C}$ complete, K small

$\underline{\mathrm{const} : \mathcal{C} \to \mathrm{Fun}(K, \mathcal{C})}$ has a left adjoint

preserves colimits as well + $\mathcal{C}$ is presentable
$\mathrm{Fun}(K, \mathcal{C})$ presentable $\Big\}$ AFT $\Rightarrow$

Presentable ∞-categories. (HTT) Ch5

Def. An ∞-cat $\mathcal{C}$ is <u>presentable</u> if $\mathcal{C}$ is cocomplete and <u>accessible</u>.

("small" in some sense).

- adjoint functor theorem
- tensor product in ∞-cat of presentable ∞-cats
- nice enough to compare with model categories.

$\boxed{\text{A.3.7.6 in HTT}}$

Thm. If $f: \mathcal{C} \to \mathcal{D}$ between presentable ∞-categories. then f has a right adjoint if and only if f preserves colimits.

Thm. If $f: \mathcal{C} \to \mathcal{D}$ between presentable ∞-categories then f has a left adjoint if and only if f is <u>accessible</u> and preserves limits.

"preserves κ-filtered colimits"

[Mar] a stronger result is proved.

HTT, also a proof.

$\mathcal{S}$: presentable.

Sp: presentable.

How do we study presentable categories.

In Sec 5.5 of HTT.

- $\mathcal{C}$ is presentable $\Leftarrow$ $\underset{\text{accessible}}{\underset{\sim}{\text{acc}}}$ $\simeq \mathrm{Ind}_\kappa(\mathcal{C}^\circ)$ Yoneda

$\Rightarrow$ accessible. $\Updownarrow$

- $\mathcal{C} \simeq$ localization of some presheaf category $\mathcal{P}(\mathcal{D})$.

(accessible) for some small ∞-cat $\mathcal{D}$.

Def. $f: \mathcal{C} \to \mathcal{D}$ is <u>a localization</u> if it has a fully faithful (accessible) right adjoint.

Fact accessible localization of presentable cat is presentable

$\rightsquigarrow$ study accessible localization of $\mathcal{P}(\mathcal{D})$.
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $
$\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{Fun}(\mathcal{D}^{op}, \mathcal{S})$.

HTT

acc localization of presentable $\mathcal{C}$. $\xleftrightarrow{1:1}$ "strongly saturated small generated <u>classes</u> of morphisms in $\mathcal{C}$.

Idea. $\underline{\underline{\mathcal{C}}}$

S be a ~~small~~ set of morphisms in $\mathcal{C}$.

$\rightsquigarrow \underline{\underline{\bar{S}}} \supseteq S$

strong saturated small generated.

Form $\mathcal{C} \xrightarrow{L} S^{-1}\mathcal{C}$

image. S-local objects in $\mathcal{C}$.