# Stability, stabilization, tensor product and smash product

- stable $\infty$-cat
- stabilization $\mathcal{C} \rightsquigarrow Sp(\mathcal{C})$
- smash product for $Sp(S)$.

① stable $\infty$-cat

- like abelian cat
- $\mathcal{C}$ stable $\infty$-cat, $h\mathcal{C}$ is a triangulated cat.

**Def.** $\mathcal{C}$: $\infty$-cat.

$0 \in \mathcal{C}$ is a zero object : initial + final.

$\mathcal{C}$ is pointed : $\mathcal{C}$ has a zero object.

$x, y \in \mathcal{C}$, $x \to 0 \to y$. zero morphism.

**Def.** $\mathcal{C}$: pointed $\infty$-cat.

triangle in $\mathcal{C}$ :

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd} \qquad (\Delta^1 \times \Delta^1 \to \mathcal{C}).$$

$(*)$

zero object

fiber seq : $(*)$ is a pullback square

cofiber seq : $(*)$ is a pushout square.

$f: X \to Y$ a morphism.

cofiber of $f$.

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$$ ← cofiber of $f$

dually, we can define fiber of a morphism.

Def. $\mathcal{C}$ is stable.

(1). $\mathcal{C}$ is pointed

(2) every morphism has fiber and cofiber

(3) a triangle is a fiber seq iff a cofiber seq.

Ex ① (we will see). $\mathcal{C}$ is $\infty$-cat w/ finite limits, the $Sp(\mathcal{C})$ is stable.

② $\mathcal{A}$ is an abelian category.

$\mathcal{D}(\mathcal{A})$ derived cat as $\infty$-cat. stable.

Thm. $\mathcal{C}$ is stable. then $h\mathcal{C}$ is a triangulated cat.

recall. $\mathcal{C}$ is triangulated.

- $\Sigma: \mathcal{C} \to \mathcal{C}$
- collection of "distinguished $\Delta$" $X \to Y \to Z \to \Sigma X$
- $\mathcal{C}$. is additive

+ Axioms ——————.

Input: $\mathcal{C}$ stable $\infty$-cat.

Jacob Lurie
Higher Algebra.
§1.1, §1.4

{ stable $\infty$-cat
stablization.

(0) If $\mathcal{C}$ is pointed $\infty$-cat w/ cofiber

from lifting properties

$$\Sigma : \mathcal{C} \to \mathcal{C}$$

$$\begin{tikzcd} \mathcal{C} \arrow[d] \\ \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C}) \arrow[d] \\ \mathcal{C} \end{tikzcd}$$

(dually, $\mathcal{C}$ w/ fiber, you can construct loop functor).

FACT: $\Sigma \dashv \Omega$ (if $\mathcal{C}$ pointed, w/ fiber + cofiber)

If $\mathcal{C}$ is stable, $\Sigma, \Omega$ are equivalences of $\infty$-cat

By adjunction, $\pi_0 \mathrm{Map}_{\mathcal{C}}(x, y)$ is an abelian group.

Also finite coproduct and product are defined

} $h\mathcal{C}$ is additive

$$\left( \begin{tikzcd} X[-1] \arrow[r] \arrow[d] & Y \arrow[d] \\ * \arrow[r] & X \amalg Y \end{tikzcd} \right)$$

Distinguished $\Delta$:

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \quad \text{in } h\mathcal{C}$$

s.t.

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & Z \arrow[r] & W \arrow[r, phantom, "\simeq"] & \Sigma X \end{tikzcd}$$

pushout pushout

Thm: $h\mathcal{C}$ is a triangulated cat

**Def** stable subcat of a stable $\infty$-cat $=$
full subcat closed under cofiber, has zero object.

**Def** exact functors. $f: \mathcal{C} \longrightarrow \mathcal{D}$

stable $\quad$ stable

$f$: it sends $0_{\mathcal{C}}$ to $0_{\mathcal{D}}$
sends fiber seq to fiber seq.

**Prop.** TFAE $\quad f: \mathcal{C} \to \mathcal{D}$

stable $\quad$ stable

(1) $f$ is exact

(2) $f$ is left exact (preserves finite limits)

(3) $f$ is right exact (preserve colimits).

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## Stabilization

Brown representability (omitted)

( Cohomology theory on $\infty$-cat $\mathcal{C}$

$$H^n : h\mathcal{C}^{op} \longrightarrow Ab.$$

$$H^n \simeq H^{n+1} \circ \Sigma \quad )$$

$H^n$ will be represented by $E(n) \in \mathcal{C}$

$$H^n \simeq H^{n+1} \circ \Sigma \quad \rightsquigarrow \quad E(n) \simeq \Omega E(n+1)$$

Fix $\mathcal{C}$ to be an $\infty$-cat with finite limits.

Def. $F : \mathcal{C} \to \mathcal{D}$

- $F$ is excisive: if $\mathcal{C}$ has pushout, $F$ sends pushout to pullback in $\mathcal{D}$.
- $F$ is reduced: if $\mathcal{C}$ has final object, $F$ sends final object to final object.

$Exc(\mathcal{C}, \mathcal{D})$ : excisive

$Fun_*(\mathcal{C}, \mathcal{D})$ : reduced

$Exc_*(\mathcal{C}, \mathcal{D})$ : excisive reduced functors.

Trick: If $\mathcal{C}$ is stable, then $F \in Exc_*(\mathcal{C}, \mathcal{D})$ iff $F$ is left exact.

If $\mathcal{C}, \mathcal{D}$ are stable, then $F \in Exc_*(\mathcal{C}, \mathcal{D})$ iff $F$ is exact.

FACT. If $\mathcal{D}$ is presentable, then $Exc$, $Exc_*$, $Fun_*$ are presentable.

Def. $S^{fin}$ : full subcategory of spaces generated by $* \in S$ under finite colimits.

$S^{fin}_{*/}$ : coslice category of $S^{fin}$.

$Sp(\mathcal{C}) := Exc_*(S^{fin}_{*/}, \mathcal{C})$.

**Thm.** $Sp(\mathcal{C})$ is a stable $\infty$-cat

(that's why stabilization).

**sketch.** ① $Exc_*(S_*^{fin}, \mathcal{C})$ is pointed and has finite limits.

② $Exc_*(S_*^{fin}, \mathcal{C})$ is stable if $\mathcal{C}$ is presentable.

③ If $\mathcal{C}$ is not presentable, $\mathcal{C} \to P(\mathcal{C})$

$$Exc_*(S_*^{fin}, \mathcal{C}) \to \underbrace{Exc_*(S_*^{fin}, P(\mathcal{C}))}_{\text{stable}}.$$

(closed under cofiber, suspension) $\square$

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$\Omega^\infty : Sp(\mathcal{C}) \to \mathcal{C}$ evaluate at $S^0$.

$(\Omega^{\infty-n} \quad S^n)$

**Prop.** TFAE

(1) $\mathcal{D}$ is stable (2). $Sp(\mathcal{D}) \to \mathcal{D}$ is an equivalence.

**Prop.** $Exc_*(\mathcal{C}, Sp(\mathcal{D})) \simeq Exc_*(\mathcal{C}, \mathcal{D})$.

$$\begin{array}{c} Exc_*(\mathcal{C}, Sp(\mathcal{D})) \\ \wr\! \mid \\ Sp(Exc_*(\mathcal{C}, \mathcal{D})) \end{array} \xrightarrow{\simeq} Exc_*(\mathcal{C}, \mathcal{D})$$

stable (if $\mathcal{C}$ has finite colimit, $\mathcal{D}$ has finite limit)

**Prop.** (HA 1.4.2.27).

If $\mathcal{C}$ is a pointed $\infty$-cat. Then TFAE.

1. $\mathcal{C}$ stable
2. $\mathcal{C}$ has finite colimit, and $\Sigma$ is an equiv
3. $\mathcal{C}$ has finite limit, and $\Omega$ is an equivalence.

(3) $\Rightarrow$ (1). $\underline{Sp(\mathcal{C})}$ stable

$$\begin{array}{c} \parallel \\ \cdots \longrightarrow X \xrightarrow{\Omega} X \xrightarrow{\Omega} X \longrightarrow \cdots \end{array}$$

$\rightsquigarrow$ $Sp(\mathcal{C}) \longrightarrow \mathcal{C}$ is an equivalence.

$\rightsquigarrow$ $\mathcal{C}$ is stable $\square$

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$Sp = Sp(S_*)$

<ins>FACT</ins>. $hSp$ is equivalent to stable htpy cat.

<ins>FACT</ins>. $Sp$ has a <ins>t-structure</ins>

$\underline{Sp^{\heartsuit}} \simeq N(Ab)$.

# Presentable stable $\infty$-categories

<u>Prop</u>. $\mathcal{C}, \mathcal{D}$ are presentable, $\mathcal{D}$ stable. then

- $Sp(\mathcal{C})$ is presentable
- $\Omega^\infty : Sp(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint

  $\Sigma^\infty_+ : \mathcal{C} \to Sp(\mathcal{C})$. (AFT)
- $\mathcal{D} \to Sp(\mathcal{C})$ has a left adjoint iff

  $\mathcal{D} \to \mathcal{C}$ has a left adjoint
- $$\mathrm{LFun}(Sp(\mathcal{C}), \mathcal{D}) \xrightarrow{\simeq} \mathrm{LFun}(\mathcal{C}, \mathcal{D}).$$

  'functors has right adjoint.

$S := \Sigma^\infty_+(*)$.

<u>Application</u>. Let $S$ be the sphere spectrum.

then

$$\begin{tikzcd} \mathrm{LFun}(Sp, \mathcal{D}) \arrow[r, "\simeq"] \arrow[dr, "\simeq"] & \mathcal{D} \arrow[r, "\simeq"] & \mathrm{Fun}(*, \mathcal{D}) \\ & \mathrm{LFun}(S*, \mathcal{D}) \arrow[u, "\simeq"] \arrow[ur, "\simeq"] & \end{tikzcd}$$

Category of spectra is freely generated under colimits by the sphere spectrum.

<u>Enhancement</u>. $\mathcal{C}$ is presentable stable $\infty$-cat

$\mathcal{C}$ is equiv to a left exact accessible localization of $\mathrm{Fun}(\mathcal{E}, Sp)$.

Tensor product $\longrightarrow$ Smash product of Sp

Think about the $\infty$-category of presentable $\infty$-cats with left adjoint functors. $\underbrace{\mathrm{Cat}_\infty^{Pr, L}}$ : morphisms are left adjoint functors btwn presentable categories

We can define a tensor product:

$$\mathcal{C}_1 \otimes \mathcal{C}_2 := \mathrm{RFun}(\mathcal{C}_1^{op}, \mathcal{C}_2)$$

preserves colimit in each component

preserves limits.

Prop. $\mathrm{BiLFun}(\mathcal{C}_1 \times \mathcal{C}_2, \mathcal{D}) \simeq \underline{\mathrm{LFun}(\mathcal{C}_1 \otimes \mathcal{C}_2, \mathcal{D})}$.

Pf adjunctions. $\square$

( Reference: A short course on $\infty$-cats. by Groth. )

$\Rightarrow$

$$\mathcal{C}_1 \times \mathcal{C}_2 \longrightarrow \mathcal{C}_1 \otimes \mathcal{C}_2$$

$\mathcal{S}$: cat of spaces.

Go to $\mathrm{Cat}_\infty^{Pr, st, L}$, tensor product.

$\underline{\underline{Sp}}$: unit for the monoidal structure

b/c $\mathcal{C} \otimes Sp \simeq Sp(\mathcal{C})$ $\longrightarrow$ Sp has a monoidal str, uniquely defined