Derived categories, triangulated categories & t-structures

I. Derived cats & triangulated cats (classical setting)

$\mathcal{A}$: Abelian cat

$Ch(\mathcal{A})$: cat of cochains in $\mathcal{A}$

obj : $X^\bullet : \cdots \to X^{-1} \to X^0 \to X^1 \to X^2 \to \cdots$

cohomology : $H^\bullet(X^\bullet)$

quasi-isomorphism : $f^\bullet : X^\bullet \to Y^\bullet$ is a quasi-iso if $H^\bullet(f^\bullet)$ is an iso.

$$Ch(\mathcal{A}) \xrightarrow{\text{localize at quasi-iso}} D(\mathcal{A})$$

derived cat of $\mathcal{A}$

variations $Ch^+(\mathcal{A})$ $Ch^-(\mathcal{A})$ $Ch^b(\mathcal{A})$

bounded from below above both sides

SH : stable homotopy category

$\mathrm{Spec} \xrightarrow{\text{localise at stable w.e.}} SH$

triangulated categories

(abbr. Δ cat)

a ~~tool~~ to describe the structure inherent in $D(A)$ or SH

$\mathcal{D}$: additive category

Def (shift functor) translation

$[1]$ / $\Sigma : \mathcal{D} \to \mathcal{D}$ an additive automorphism.

Def (distinguished triangles) exact

$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \quad \in \mathcal{D}$

Def (Δ cat)

1) an additive cat $\mathcal{D}$ with a shift functor Σ

2) a class of exact Δ in $\mathcal{D}$

satisfying the following axioms.

TR1

1) (identity) $X \xrightarrow{id} X \to 0 \to$ is a $\triangle$
2) (completion) $\forall\, X \xrightarrow{f} Y$, $\exists\, Z$ fitting into a $\triangle$: $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$
3) $\{\triangle\}$ are closed under iso

TR2 (rotation)

If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is a $\triangle$

then $Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{\Sigma f} \Sigma Y$ are $\triangle$.

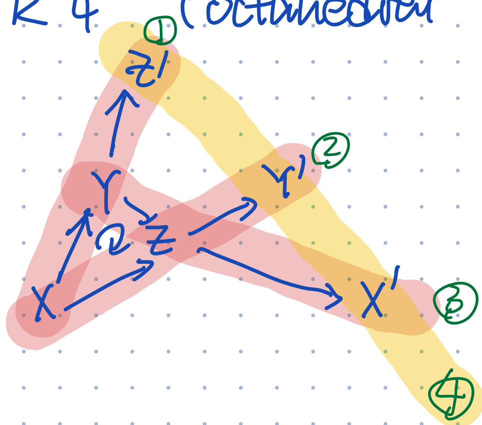
$\Sigma^{-1} Z \xrightarrow{\Sigma^{-1} h} X \xrightarrow{f} Y \xrightarrow{g} Z$

TR3 (weak functoriality)

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "u"'] \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[r] \arrow[d, "v"] & Z \arrow[r] \arrow[d, dashed, "w"] & \Sigma X \arrow[d, "\Sigma u"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X'
\end{tikzcd}$$

are $\triangle$, commutes $\Longrightarrow$ $\exists$ w s.t. everything $\circlearrowright$.

TR 4 (octahedral axiom)



If ① ② ③ are $\triangle$
$\Longrightarrow$ have ④ and it's $\triangle$.
and everything $\circlearrowright$.

Example of Δ cats :

$D(A)$, $D^{\pm,b}(A)$, SH

Σ : shift | Σ : $\wedge S^1$

Δ: exact sequences | Δ : cofiber / fiber sequences

propeties

1) $X \to Y \to Z \to \Sigma X$ a Δ

$$\begin{tikzcd} X \arrow[r] \arrow[rr, bend right, "=0"'] & Y \arrow[r] \arrow[rr, bend left, "=0"] & Z \arrow[r] & \Sigma X \end{tikzcd}$$

2) $\mathrm{Hom}_D(E,-)$ takes Δ to a LES.

exercises: prove 1) 2).

(using 2)) prove that in TR1 2), Z is unique up to isomorphism.

warning Z is unique up to nonunique iso.

TR 3 : $\dashrightarrow$ is not unique.

$$\begin{tikzcd} X \arrow[r] \arrow[d, "u"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[r] \arrow[d, "v"] & Z \arrow[r] \arrow[d, dashed, "w"] & \Sigma X \arrow[d, "\Sigma u"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X' \end{tikzcd}$$

Ex

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & 0 \arrow[r] \arrow[d] & \Sigma A \arrow[r] \arrow[d, dashed] & \Sigma A \arrow[d] \\ 0 \arrow[r] & \Sigma A \arrow[r] & \Sigma A \arrow[r] & 0 \end{tikzcd}$$

weak functoriality

for a stable ∞-cat $\mathcal{D}$

① hD is a Δ category

② cof in D is essentially unique.

II Derived categories (higher categorical)

Ref Lurie Higher Algebra Section 1.3.

A : Abelian cat

$D^-(A)$: ∞-cat of (bdd) derived cat of A

roughly: obj: proj & right bdd chain cplx

1-morph: maps of chain cplx

2-morph: chain homotopies

$\vdots$ $\qquad$ $\vdots$

cohom $\to X^n \to X^{n+1} \to X^{n+2} \to$ $\qquad$ ~~left~~ right bdd

hom $\to X_n \to X_{n-1} \to X_{n-2} \to$

Sketch construction:

$A \rightsquigarrow A_{proj}$: full sub of proj obj

$\rightsquigarrow Ch^-(A_{proj})$ right bdd chain cplx

$\rightsquigarrow$ make $Ch^-(A_{proj})$ enriched over $Ch(Ab)$

$$\mathrm{Ch}(\mathrm{Ab}) \xrightarrow{\geq 0} \mathrm{Ch}(\mathrm{Ab})_{\geq 0} \xrightarrow{DK} \mathrm{Ab}_\Delta \xrightarrow{U} \mathrm{Set}_\Delta$$

$\rightsquigarrow$ $\mathrm{Ch}^-(A_{proj})$ enriched over Set_Δ

$\rightsquigarrow$ $N : \mathrm{Cat}_\Delta \to \mathrm{Set}_\Delta$

$$\mathrm{Ch}^-(A_{proj}) \longmapsto \mathcal{D}^-(A)$$

Rmk : In HA its another construction N_{dg}, but equivalent.

Properties

1. $h\mathcal{D}^-(A)$ is classical bold derived cat.
2. $\mathcal{D}^-(A)$ is stable ∞-cat

III. t-str. on Δ cats

$\mathcal{D}$: Δ cat

Def (t-str)

1) a pair of full sub cats $(\mathcal{D}_{\geq 0}, \mathcal{D}_{\leq 0})$, closed under iso

2) $\Sigma \mathcal{D}_{\geq 0} \subseteq \mathcal{D}_{\geq 0}$, $\Sigma^{-1}\mathcal{D}_{\leq 0} \subseteq \mathcal{D}_{\leq 0}$

3) $\forall X \in \mathcal{D}_{\geq 0}$, $\forall Y \in \mathcal{D}_{\leq 0}$
$\mathrm{Hom}_{\mathcal{D}}(X, \Sigma^{-1} Y) = 0$.

4) $\forall Z \in \mathcal{D}$, $\exists$ a Δ

$$\underset{\mathcal{D}_{\geq 0}}{\underset{\cap}{X}} \longrightarrow Z \longrightarrow \underset{\Sigma^{-1}\mathcal{D}_{\leq 0}}{\underset{\cap}{Y}} \longrightarrow$$

Notations

$$\mathcal{D}_{\geq n} := \Sigma^{n} \mathcal{D}_{\geq 0}$$
$$\mathcal{D}_{\leq n} := \Sigma^{-n} \mathcal{D}_{\leq 0}$$
$$\mathcal{D}^{\heartsuit} := \mathcal{D}_{\geq 0} \cap \mathcal{D}_{\leq 0}$$

Examples

1) $D(A)$

$D(A)_{\geq 0} = \{X \mid H_i(X)$ concentrates in $i \geq 0\}$
$D(A)_{\leq 0} = \{X \mid$ ——— " ——— $\leq 0\}$

$$D(A)^{\heartsuit} = A$$

2) SH (Postnikov t-str.)

$SH_{\geq 0} = \{X \mid \pi_i(X)$ concentrates in $i \geq 0\}$
$SH_{\leq 0} = \{X \mid$ ——— " ——— $i \leq 0\}$

$SH^{\heartsuit} = Ab$

truncation functor

$\tau_{\geq n}$: colocalization wrt $\mathcal{D}_{\geq n}$

$\tau_{\leq n}$: localization wrt $\mathcal{D}_{\leq n}$

In Ex 2) $\tau_{\geq n} \rightsquigarrow$ n-th connective cover

$\tau_{\leq n} \rightsquigarrow$ n-th coconnective cover.

Application : get filtered objs

$\cdots \to \tau_{\geq n} X \to \tau_{\geq n-1} X \to \tau_{\geq n-1} X \to \cdots$

use it to define SS.

In Ex 2) : gives the Postnikov tower

In the stable ∞-cat setting

t-str. ~~on~~ on stable ∞-cat is defined by passing to the homotopy category.

t-str on $D^-(A)$

$(D^-(A)_{\geq 0}, D^-(A)_{\leq 0})$ defined as in Ex 1)

Prop $\quad D^-(A)^{\heartsuit} = N(A)$

(universal property)

$\mathcal{A}$: Abelian cat, with $\overset{\text{enough}}{\text{proj}}$

$\mathcal{C}$: stable ∞-cat, with t-str. left complt

nothing in $\mathcal{C}_{\geq\infty}$

$\mathcal{E} \subseteq \mathrm{Fun}(\mathcal{D}^-(\mathcal{A}), \mathcal{C})$ full sub cat spanned by right t-exact functors which sends proj in $\mathcal{A}$ to $\mathcal{C}^{\heartsuit}$.

preserve ≥ 0

$\mathcal{D}^-(\mathcal{A}) \to \mathcal{C}$.

$\mathcal{D}^-(\mathcal{A})_{\geq 0} \to \mathcal{C}_{\geq 0}$

$\mathcal{D}^-(\mathcal{A})^{\heartsuit} \not\to \mathcal{C}^{\heartsuit}$

$\searrow \mathcal{C}_{\geq 0}$

$$\Longrightarrow \quad \mathcal{E} \simeq \mathrm{Fun}^{\text{right exact}}(\mathcal{A}, \mathcal{C}^{\heartsuit})$$
$$F \longmapsto \tau_{\leq 0}(F|_{\mathcal{A}})$$

Compare it to the classical result:

$F : \mathcal{A} \to \mathcal{B}$ right exact functor of Ab cats

$\mathcal{A}$ has enough projectives

$\rightsquigarrow$ can define $\underbrace{LF}_{\text{left derived functor}} : D\mathcal{A} \to D\mathcal{B}$