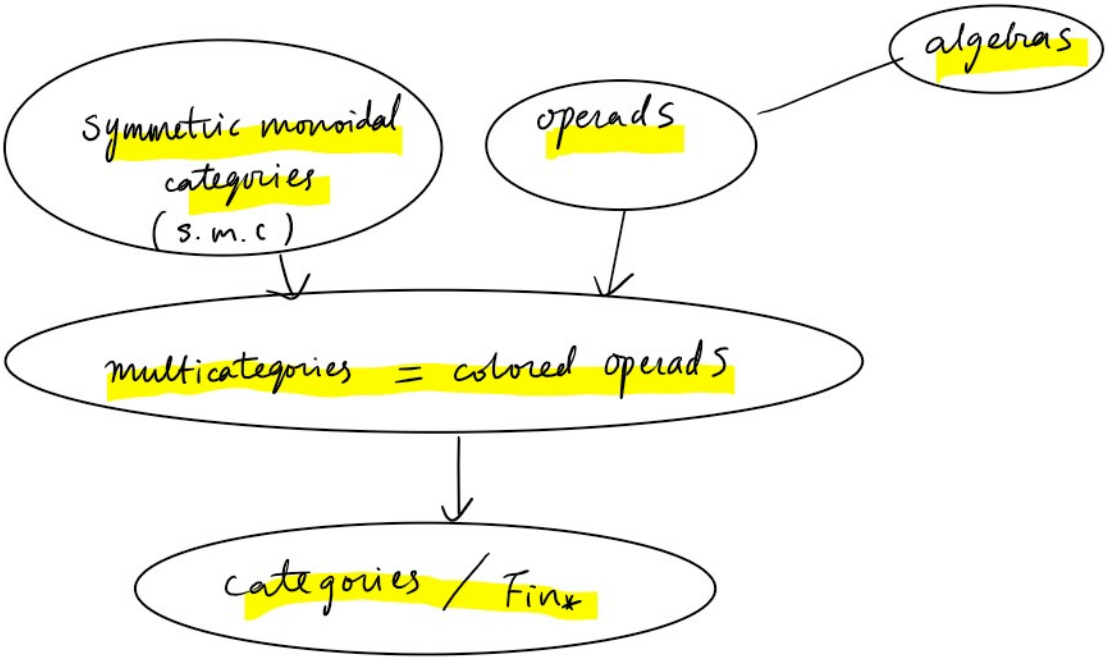
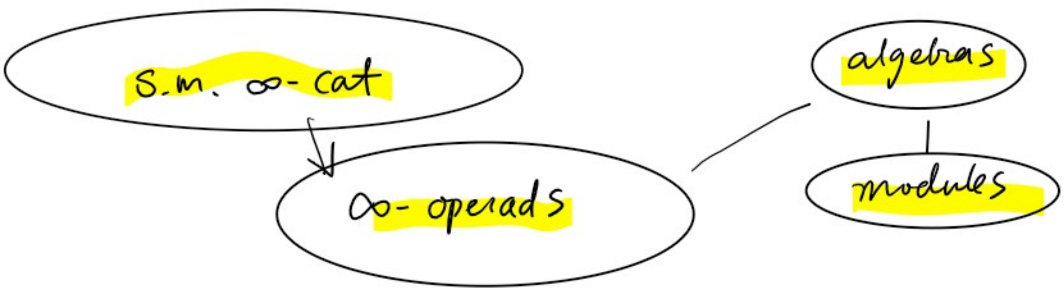
Monoidal structures, part 1





§0. recall $p: X \to S$

inner fib $\Leftarrow$ cocartesian fib $\Leftarrow$ left fib $\Leftarrow$ kan fib

- inner fib: X_s is an ∞-cat
- cocartesian fib: $f: s \to T$, have $f_!: X_s \to X_T$ up to canonical equi
- left fib: X_s is a kan complex
- $\Leftarrow$ kan fib: same over a point

slogan.

- cocart fib/S $\simeq \mathrm{Fun}(S, \mathrm{Cat}_\infty)$
- cart fib/S $\simeq \mathrm{Fun}(S^{op}, \mathrm{Cat}_\infty)$
- left fib/S $\simeq \mathrm{Fun}(S, \mathrm{Kan})$
- right fib/S $\simeq \mathrm{Fun}(S^{op}, \mathrm{Kan})$

§1. symmetric monoidal category $(\mathcal{C}, \otimes, I)$

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Baez, some definitions everybody should know
https://math.ucr.edu/home/baez/qg-fall2004/definitions.pdf

Data: $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

$I \in ob(\mathcal{C})$

natural isomorphisms
$$\begin{cases} \alpha_X : I \otimes X \simeq X & \text{(left) unitor} \\ \beta_{X,Y} : X \otimes Y \simeq Y \otimes X & \text{commutator} \\ \gamma_{X,Y,Z} : (X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z) & \text{associator} \end{cases}$$

Requirement: coherent diagrams,

including pentagon and hexagon diagrams

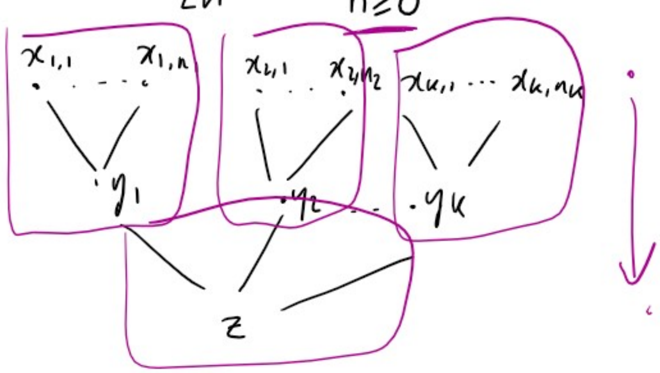
Examples:
- $(\mathrm{Vect}_k, \otimes, k)$
- $(\mathrm{Space}, \times, *)$
- $(\mathrm{Sp}, \wedge, \mathbb{S})$

Multicategory $\mathcal{M}$

Objects. • x, y

multimorphisms. • $\mathrm{Mul}(x_1, \cdots, x_n; y)$ $\circlearrowleft \Sigma_n$, $n \geq 0$

compositions •



Example. $\mathrm{Mul}(V_1, V_2; W) = \{\text{bilinear maps } V_1 \times V_2 \to W\}$

Example. $\mathrm{Mul}(V_1, V_2; W) = \{\text{bilinear maps } V_1 \times V_2 \to W\}$

$= \{\text{linear maps } V_1 \otimes V_2 \to W\}$

This is general:

$$\text{s.m. C} \longrightarrow \text{multi-cat}$$
$$(\mathcal{C}, \otimes, I) \qquad \mathcal{C}^{\otimes}$$

What if we consider $(x_1, \cdots, x_n)$ as a single object?

$$\text{multi-cat} \longrightarrow \text{categories}/\mathrm{Fin}_*.$$

$$\mathcal{M} \longmapsto p : \mathcal{M}^{\otimes} \to \mathrm{Fin}_*$$

$\mathcal{M}^{\otimes}$ obj $(x_1, \cdots, x_n)$, $n \geq 0$, $x_i \in \mathcal{M}$

mor $(x_1, \cdots, x_n) \to (y_1, \cdots, y_m)$

consists of

$$\begin{cases} \alpha : \{0, 1, \cdots, n\} \to \{0, 1, \cdots, m\} \\ f_j \in \mathrm{Mul}(x_{\alpha^{-1}(j)}; y_j) \\ \quad 1 \leq j \leq m \end{cases}$$

Fin_* : obj : $\langle n \rangle = \{0, 1, \cdots, n\}$

mor : based maps

Why introduce the base point 0?

$\alpha : \langle n \rangle \to \langle m \rangle$ can "forget" elements in $\langle n \rangle$.

- Important to forget points.

Question:

When and how to recover $(\mathcal{C}, \otimes, I)$ from $p : \mathcal{C}^{\otimes} \to \mathrm{Fin}_*$?

Answer:

$$\left\{ \text{s.m.c up to symmetric monoidal equivalence} \right\}$$

$$\Updownarrow$$

cocartesian condition   segal condition

$$\left\{ p : \mathcal{O} \to \mathrm{Fin}_* \text{ satisfying (M1) and (M2)} \right\}$$

(M1) p is an op-fibration of categories

meaning that $\forall \alpha$, and C $\exists f$, such that $\forall C''$, and g, β with $p(g) = \beta \circ \alpha$ $\exists !\, h$ with $p(h) = \beta$ and $h \circ f = g$.

f is "p-coCartesian"

$$\begin{tikzcd}
\mathcal{C}^{\otimes} \arrow[d, "p"] & C \arrow[rr, "g"] \arrow[dr, "f"'] & & C'' \\
& & C' \arrow[ur, "h"'] & \\
\mathrm{Fin}_* & \langle n \rangle \arrow[rr, "p(g)"] \arrow[dr, "\alpha"'] & & \langle l \rangle \\
& & \langle m \rangle \arrow[ur, "\beta"'] &
\end{tikzcd}$$

content of (M1) for $p : \mathcal{O} \to \mathrm{Fin}_*$

$\alpha : \langle n \rangle \to \langle m \rangle$ in Fin_*

fibers over $\langle n \rangle$

$\Rightarrow$ $\alpha_! : \mathcal{O}_{\langle n \rangle} \to \mathcal{O}_{\langle m \rangle}$ well defined up to canonical isomorphism

Verification for $\mathcal{O} = \mathcal{C}^{\otimes}$:

$\mathcal{C}^{\otimes}_{\langle n \rangle} = \mathcal{C}^n$ $\alpha : \langle n \rangle \to \langle m \rangle$ $\big\langle$ projections to 1, injections

$\alpha_!$ induced by $\mathcal{C}^k \to \mathcal{C}$

$$(x_1, \cdots, x_k) \mapsto x_1 \otimes x_2 \otimes \cdots \otimes x_k$$

and $* \to \mathcal{C}$

$$* \mapsto I$$

(M2) Segal condition

- $\rho^i : \langle n \rangle \longrightarrow \langle 1 \rangle$

  $i \longmapsto 1$

  $\text{o.w} \longmapsto 0$

- $\mathcal{D}_{\langle n \rangle} \xrightarrow[\sim]{(\rho^1_!, \cdots, \rho^n_!)} \mathcal{D}^n_{\langle 1 \rangle}$

$\mathcal{C}^n = \mathcal{C}^n$

Verification for $\mathcal{D} = \mathcal{C}^{\otimes}$:

$$\rho^i_! : \mathcal{C}^{\otimes}_{\langle n \rangle} = \mathcal{C}^n \longrightarrow \mathcal{C}^{\otimes}_{\langle 1 \rangle} = \mathcal{C}$$

$$(x_1, \cdots, x_n) \longmapsto x_i$$

content of (M2) :

- $\mathcal{D}_{\langle n \rangle} \xrightarrow{\sim} \mathcal{D}^n_{\langle 1 \rangle}$

not equal, but equivalent by built in data

To recover from $p : \mathcal{D} \to \mathrm{Fin}_*$ the s.m.c. $\mathcal{C}$

(M2) $\Rightarrow$. $\mathcal{C} = \mathcal{D}_{\langle 1 \rangle}$

. $\mathcal{D}_{\langle 0 \rangle} \simeq *$ $\quad \langle 0 \rangle \to \langle 1 \rangle \Rightarrow I \to \mathcal{C}$

. $\alpha : \langle 2 \rangle \to \langle 1 \rangle \quad 1, 2 \mapsto 1$

$\Rightarrow \otimes : \mathcal{D}^2_{\langle 1 \rangle} \xleftarrow[(M2)]{\sim} \mathcal{D}_{\langle 2 \rangle} \xrightarrow{\alpha_!} \mathcal{D}_{\langle 1 \rangle}$

. $\beta_{x,y}$ is given by $\sigma : \langle 2 \rangle \to \langle 2 \rangle$

. . . .

Summary

	classical	new
Data	• $\mathcal{C}$ • $\otimes$, I • α_X • $\beta_{X,Y}$ • $\gamma_{X,Y,Z}$	$p: \mathcal{O} \to Fin_*$
Requirement	• diagrams commute (e.g. MacLane pentagons)	(M1) op-fibration → coCartesian fibration (M2) segal

remark
{ active
inert •

Advantage of the new perspective:

① simpler data

② easier to generalize to ∞-categories

generalization of (M1)

Def. A symmetric monoidal ∞-category is a coCartesian fibration $p: \mathcal{C}^{\otimes} \to N(Fin_*)$ such that $(*)$ $\rho_!$ induced by ρ^i determines an equivalence $\mathcal{C}^{\otimes}_{\langle n \rangle} \simeq (\mathcal{C}^{\otimes}_{\langle 1 \rangle})^n$.

(M2)

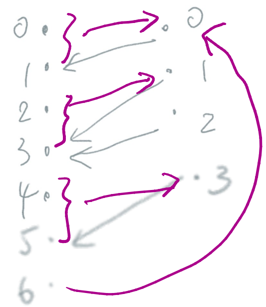
Homework:

① Is $id: N(Fin_*) \to N(Fin_*)$ a symmetric monoidal ∞-category?

② $\Delta^{op} \to Fin_*$ (Segal's Γ-space in categories and cohomology theories) obselete
$[m] \mapsto \langle m \rangle$

$[m] \longmapsto \langle m \rangle$ (categories and cohomology theories)

$([m] \leftarrow [n]) \longmapsto (\langle m \rangle \to \langle n \rangle)$ (In fact, this is s!)

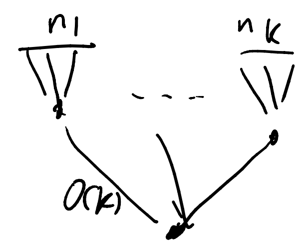


Is $N(\Delta^{op}) \to N(Fin_*)$ a symmetric monoidal ∞-category? An ∞-operad? (No)

§2. Operads

data: Sets $\mathcal{O}(n) \circlearrowleft \Sigma_n$, $n \geq 0$

- $\eta: * \to \mathcal{O}(1)$ parametrizes $id: A \to A$
- $\gamma: \mathcal{O}(k) \times \mathcal{O}(n_1) \times \cdots \mathcal{O}(n_k) \longrightarrow \mathcal{O}(n_1 + \cdots + n_k)$



requirement: unital, equivariance, associativity diagrams.

Quick observation:

- an operad = a multi-category with one object.

  $\mathcal{O}(n) = Mul(\underbrace{*, \dots, *}_{n}; *)$ including $n = 0$

$\Rightarrow$ can form $p: \mathcal{O}^{\otimes} \to Fin_*$

$$\begin{cases} \text{obj}: \langle n \rangle = (\underbrace{*, \dots, *}_{n}) & n \geq 0 \\ \text{mor}: \alpha: \langle m \rangle \to \langle n \rangle \text{ and } \{x_j \in \mathcal{O}(|\alpha^{-1}(j)|) \mid 1 \leq j \leq n\} \end{cases}$$

Algebra

Examples:

Triv $\quad \mathrm{Triv}(n) = \begin{cases} * & n = 1 \\ \emptyset & n \neq 1 \end{cases}$

everything

Triv $\quad \mathrm{Triv}(n) = \begin{cases} \ast \\ \phi & n \neq 1 \end{cases}$ $\quad$ everything

$\mathrm{Triv}^{\otimes}$ obj : $\langle n \rangle$

mor : $\sigma \in \Sigma_n$

E_0 $\quad E_0(n) = \begin{cases} \ast & n = 0 \text{ or } 1 \\ \phi & \text{o.w.} \end{cases}$ $\quad$ based objects

$E_0^{\otimes}$ obj : $\langle n \rangle$ (based)

mor : injections

Comm $\quad \mathrm{Comm}(n) = \ast$ $\quad$ comm monoids

$\mathrm{Comm}^{\otimes}$ obj : $\langle n \rangle$ $\quad \mathrm{Comm}^{\otimes} = \mathrm{Fin}_{\ast}$

mor : based maps

Ass $\quad \mathrm{Ass}(n) = \Sigma_n$ $\quad$ (associative) monoids

$\mathrm{Ass}^{\otimes}$ skipped

Algebra over $\mathcal{O}$ in $(\mathcal{C}, \otimes, I)$:

- $A \in \mathcal{C}$ with
- data : $\mathcal{O}(n) \underset{\Sigma_n}{\times} A^{\otimes n} \longrightarrow A$
- requirement : unital and associativity diagrams

content : $\mathcal{O}(n)$ parametrizes n-ary operations on A.

If we allow $\mathcal{O}(n)$ to be spaces, we can parametrize "unique n-ary operations up to contractible choices".

Remark : We have seen the need for coherence data already in a symmetric monoidal category : there's

$\{ I : \ast \to \mathcal{C}$ $\quad$ look like $\quad \mathcal{C}^0 \to \mathcal{C}^1$

$\mathcal{C}^2 \to \mathcal{C}^1$

In a symmetric monoidal category:

$$\begin{cases} I : * \to \mathcal{C} \\ \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \end{cases}$$

look like $\mathcal{C}^0 \to \mathcal{C}^1$, $\mathcal{C}^2 \to \mathcal{C}^1$

$\Rightarrow$ $\mathcal{C}$ is some sort of algebra in $(\mathrm{Cat}, \times, *)$

It's not a commutative algebra: we have coherence data α_X, $\beta_{X,Y}$, $\gamma_{X,Y,Z}$.

In a symmetric monoidal ∞-category, even more coherence data is introduced.

strictness − / coherent data + ↓	cat	∞-cat
	comm algebra	
	symmetric monoidal category	
	$\mathcal{E}_\infty$-algebra	symmetric monoidal ∞-category $\parallel$ comm algebra in s.m. ∞-cat

Fact. $\mathrm{CAlg}(\mathrm{Cat}_\infty^\times) \simeq \mathrm{Cat}_\infty^{\mathrm{sMon}}$

$\mathrm{Alg}(\mathrm{Cat}_\infty^\times) \simeq \mathrm{Cat}_\infty^{\mathrm{Mon}}$

Homework.

③ $\mathcal{C}$: symmetric monoidal category. Find out the p-cocartesian edges in $p : \mathcal{C}^\otimes \to \mathrm{Fin}_*$

④ $\mathcal{O}$: operad. Do the same for $p : \mathcal{O}^\otimes \to \mathrm{Fin}_*$

§3. ∞-operads.

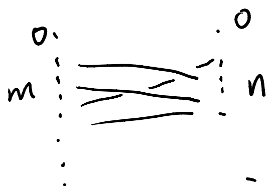
Recall

Def. A symmetric monoidal ∞-category is a cocartesian fibration $p: \mathcal{C}^{\otimes} \to N(\mathrm{Fin}_*)$ such that $(*)$ $\rho^i_!$ induced by ρ^i determines an equivalence $\mathcal{C}^{\otimes}_{\langle n \rangle} \simeq (\mathcal{C}^{\otimes}_{\langle 1 \rangle})^n$.

relax

Def. In Fin_* $\alpha: \langle m \rangle \to \langle n \rangle$ is inert (forgetful) if

$$|\alpha^{-1}\{i\}| = 1 \qquad 1 \le i \le n$$



Example. $\rho^i : \langle n \rangle \to \langle 1 \rangle$ "coordinates"

Def. An ∞-operad is $p: \mathcal{O}^{\otimes} \to N(\mathrm{Fin}_*)$ between ∞-categories such that

(1) $\forall$ inert $\alpha: \langle m \rangle \to \langle n \rangle$, $C \in \mathcal{O}^{\otimes}_{\langle m \rangle}$, $\exists$ p-cocartesian lift $\bar{\alpha}$.

(2) $\forall$ $f: \langle m \rangle \to \langle n \rangle$, $c \in \mathcal{O}^{\otimes}_{\langle m \rangle}$, $c' \in \mathcal{O}^{\otimes}_{\langle n \rangle}$, choose p-cocartesian lifts of $\rho^i: \langle n \rangle \to \langle 1 \rangle$

$\overline{\rho^i} : c' \to c'_i \in \mathcal{O}^{\otimes}_{\langle 1 \rangle}$

Then

component over f component over $\rho^i \circ f$

$$\mathrm{Map}^f(c, c') \xrightarrow{(\overline{\rho^i} \circ -)_{i=1}^n} \prod^n \mathrm{Map}^{\rho^i \circ f}(c, c'_i)$$

$$\mathrm{Map}^{f}_{\mathcal{O}^\otimes}(C, C') \xrightarrow{(\bar{p}^i\circ -)_{i=1}^{n}} \prod_{i=1}^{n} \mathrm{Map}^{p^i\circ f}_{\mathcal{O}^\otimes}(C, C'_i)$$

is a homotopy equivalence.

(3) $\forall\, C_1, \cdots, C_n \in \mathcal{O}^\otimes_{\langle 1\rangle}$, $\exists\, C \in \mathcal{O}^\otimes_{\langle n\rangle}$ and p-cocartesian morphisms $\bar{p}^i : C \to C_i$ covering $p^i : \langle n\rangle \to \langle 1\rangle$.

Rem (3') $p^i_!$ induces $\phi : \mathcal{O}^\otimes_{\langle n\rangle} \xrightarrow{\sim} (\mathcal{O}^\otimes_{\langle 1\rangle})^n \Rightarrow (3)$

Segal condition

(1) $\Rightarrow$ ϕ well defined

(2) $\Rightarrow$ ϕ is fully faithful

(3) $\Rightarrow$ ϕ is essentially surjective

What's (2)?

- $\mathcal{O}$: (ordinary) operad $\xrightarrow{\text{before}}$ $p : \mathcal{O}^\otimes \to \mathrm{Fin}_*$

  $\rightsquigarrow p : N(\mathcal{O}^\otimes) \to N(\mathrm{Fin}_*)$ ∞-operad

Verify (2): $f : \langle 4\rangle \to \langle 2\rangle \xrightarrow{p^i} \langle 1\rangle$

$0, 1 \mapsto 0$
$2 \mapsto 1$
$3, 4 \mapsto 2$

unique element in $\mathcal{O}^\otimes_{\langle 4\rangle}$

$$\begin{array}{ccc} \mathrm{Map}^{f}_{\mathcal{O}^\otimes}(4, 2) & = & \mathcal{O}(1) \times \mathcal{O}(2) \\ \downarrow \bar{p}^i & & \downarrow \text{projection} \\ \mathrm{Map}^{p^i\circ f}_{\mathcal{O}^\otimes}(4, 1) & = & \begin{cases} \mathcal{O}(1) & i=1 \\ \mathcal{O}(2) & i=2 \end{cases} \end{array}$$

Always pick $\overset{\in \mathcal{O}(1)}{\bar{p}^i} : n \to 1$

as $\eta \in \mathcal{O}(1)$.

* η is a coCartesian edge

$\mathcal{O}^{\otimes}$
$$\begin{tikzcd} n \arrow[rr, "g"] \arrow[dr, "\eta"'] & & m \\ & 1 \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

cases $\begin{cases} \cdot\ \beta(1) = 0 & \text{easy to check} \\ \cdot\ \beta \text{ is injective.} & p(g) = \beta \circ p^i = p^j \text{ for some } j. \end{cases}$

$g \leftrightarrow x \in \mathcal{O}(1)$

$h \leftrightarrow$ element y in $\underbrace{\mathcal{O}(1)}_{\text{monoid}}$

Fin_*
$$\begin{tikzcd} \langle n\rangle \arrow[rr, "p(g)"] \arrow[dr, "p^i"'] & & \langle m\rangle \\ & \langle 1\rangle \arrow[ur, "\beta"'] & \end{tikzcd}$$

want $y \circ \underbrace{\eta}_{\text{identity in } \mathcal{O}(1)} = x$

Take $y = x$.

Then $\bar{p}^i \circ -$ is projection onto the component.

(2) is saying that morphisms in $\mathcal{O}^{\otimes}$ all are made up by morphisms over $\langle n\rangle \to \langle 1\rangle$

$\mathcal{O}(n)$

Example. $\mathrm{id}: \underset{\mathrm{Comm}^{\otimes}}{\overset{\parallel}{N(\mathrm{Fin}_*)}} \to N(\mathrm{Fin}_*)$.

Example. $p: \mathcal{C}^{\otimes} \to N(\mathrm{Fin}_*)$ a symmetric monoidal ∞-cat.

$\Rightarrow p$ is also an ∞-operad.

— End —