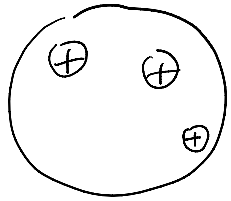
Monoidal structures, part 2

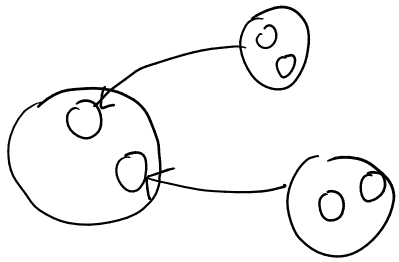
§4 E_n-operads (1 cat $\to$ ∞ cat)

Little n-disk operad $\mathcal{D}_n(k)$

<u>spaces</u>



composition



n-fold loop spaces and $\mathcal{D}_n$-algebra $= E_n$-algebra

$X \simeq \Omega^n Y \Rightarrow X$ is a $\mathcal{D}_n$-algebra

X is a $\mathcal{D}_n$-algebra $\Rightarrow \exists\ X \to \Omega^n Y$ map of $\mathcal{D}_n$-alg & group completion

$$\boxed{\begin{array}{l} \mathcal{D}_n\text{-alg}_{gp} \simeq \text{n-fold loop spaces} \\ E_\infty\text{-alg}_{gp} \simeq \infty\text{-loop spaces} \simeq \text{connective spectra} \end{array}}$$

	topology $\Omega^n X$	algebra $\pi_0(\Omega^n X) = \pi_n(X)$
$n=0$	E_0-alg = based space	set
$n=1$	E_1-alg	group
$n=2$	E_2-alg	abelian group
	$\vdots$	
$n=\infty$	E_∞-alg	

$n = \infty$ | E_∞-alg |

∞-operad $E_n^{\otimes}$

$\mathcal{O}$: simplicial operad, fibrant — $\mathcal{O}(n)$ are fibrant simplicial sets = Kan complexes.

$\Rightarrow$ $\mathcal{O}^{\otimes} \longrightarrow Fin_*$

fibrant simplicial category

$\Rightarrow$ $N(\mathcal{O}^{\otimes}) \to N(Fin_*)$ is an ∞-operad

$N(\mathcal{O}^{\otimes})$ — simplicial nerve

$\parallel$

$N^{\otimes}(\mathcal{O})$ — operadic nerve

§5. algebras (∞-cat)

Task:

Define $\mathcal{O}^{\otimes}$-algebra in a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$.

∞-operads $p : \mathcal{O}^{\otimes} \to N(Fin_*)$. $q : \mathcal{C}^{\otimes} \to N(Fin_*)$

Def. inert morphisms e in $\mathcal{O}^{\otimes}$

= p-cocartesian + $p(e)$ inert in Fin_*

In $\mathcal{O}^{\otimes}$: units $\in \mathcal{O}(1)$

$\mathcal{C}^{\otimes}$: coordinates

> forgetful functor

Def. $f : \mathcal{O}^{\otimes} \to \mathcal{C}^{\otimes}$ map of ∞-operads

if (1)

$$\begin{tikzcd} \mathcal{O}^{\otimes} \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathcal{C}^{\otimes} \arrow[dl, "q"] \\ & N(Fin_*) & \end{tikzcd}$$

(2) f sends inert morphisms in $\mathcal{O}^{\otimes}$ to inert morphisms in $\mathcal{C}^{\otimes}$.

Rem With (1), (2) $\Leftrightarrow$ (2') f preserves inert morphisms

morphisms in $\mathcal{C}$.

Rem. With (1), (2) $\Leftrightarrow$ (2') f preserves inert morphisms over $\rho^i: \langle n \rangle \to \langle 1 \rangle$.

Write $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ for the full-subcategory of $\mathrm{Fun}_{N(\mathrm{Fin}_*)}(\mathcal{O}^{\otimes}, \mathcal{C}^{\otimes})$ spanned by ∞-operad maps.

- When $\mathcal{O}^{\otimes} = \mathrm{Comm}^{\otimes} = N(\mathrm{Fin}_*) \to N(\mathrm{Fin}_*)$, write $\mathrm{CAlg}(\mathcal{C})$.

Generalization from symmetric monoidal ∞-cat to $\mathcal{O}$-monoidal ∞-cat

Prop (HA 2.1.2.12) $\mathcal{O}^{\otimes} \to N(\mathrm{Fin}_*)$ ∞-operad
$p: \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ cocartesian fib.

TFAE

(1) $q: \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes} \to N(\mathrm{Fin}_*)$ is an ∞-operad

(2) $\forall\, T \in \mathcal{O}^{\otimes}_{\langle n \rangle}$ and p-cocartesian lifts $\bar{\rho}^i: T \to T_i$, inert

$\bar{\rho}^i$ induces an equivalence of ∞-categories
$$\mathcal{C}^{\otimes}_T \to \prod_{i=1}^{n} \mathcal{C}^{\otimes}_{T_i}$$

When (1) (or equivalently (2)) is satisfied, we say p is a coCartesian fibration of ∞-operads, and that p exhibits $\mathcal{C}^{\otimes}$ as an $\mathcal{O}$-monoidal ∞-category. (p is automatically a map of ∞-operads)

($\mathcal{O}$-monoidal ∞-category = cocartesian fib $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$

(∞-operads)

$\mathcal{O}$-monoidal ∞-category = coCartesian fib $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$

Example: symmetric monoidal ∞-cat $p: \mathcal{C}^\otimes \to N(\mathrm{Fin}_*)$

= coCartesian fibration of ∞-operads p

= $\mathrm{Comm}^\otimes$-monoidal ∞-category.

Def. A map of ∞-operads $p: \mathcal{O}^\otimes \to \mathcal{C}^\otimes$ is a fibration of ∞-operads if p is a categorical fibration.

fibration in Joyal model str.

Prop (HA 2.1.2.22) $p: \mathcal{O}^\otimes \to \mathcal{C}^\otimes$ map of ∞-operads, inner fibration. TFAE

(1) p is a fibration

(2) $\forall$ $X \in \mathcal{O}^\otimes$, $C \in \mathcal{C}^\otimes$, inert morphism $p(X) \to C$

$\exists$ inert lift $X \to \overline{C}$

When (1) (or equivalently (2)) is true,

inert morphism e in $\mathcal{O}^\otimes$

= p-coCartesian and $p(e)$ inert in $\mathcal{C}^\otimes$.

Example. $p: \mathcal{O}^\otimes \to N(\mathrm{Fin}_*)$ ∞-operad

Example. $p: \mathcal{O}^{\otimes} \to N(\mathrm{Fin}_*)$ ∞-operad

$\Rightarrow$ a fibration of ∞-operads $\mathcal{O}^{\otimes} \to \mathrm{Comm}^{\otimes}$ (not a cocart-fib)

Def.

$$\begin{tikzcd} (\mathcal{O}')^{\otimes} \arrow[rr, dashed, "F"] \arrow[dr, "\text{map of } \infty\text{-operads}"'] & & \mathcal{C}^{\otimes} \arrow[dl, "\text{fibration of } \infty\text{-operads}"] \\ & \mathcal{O}^{\otimes} & \end{tikzcd}$$

$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) := \mathrm{Alg}_{\mathcal{O}/\mathcal{O}}(\mathcal{C})$ is the correct notion of $\mathcal{O}$-alg in a $\mathcal{O}$-monoidal cat.

$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) =$ ∞-cat of map of ∞-operads F

cocartesian fib of ∞-operads $\Rightarrow$ fibration of ∞-operads $\Rightarrow$ map of ∞-operads

over $\mathrm{Comm}^{\otimes}$: cocartesian fib of ∞-operads $=$ symmetric monoidal ∞-cat

fibration of ∞-operads $=$ ∞-operad

Prop. $\mathcal{C}$: symmetric monoidal category

Then $\mathrm{CAlg}(N(\mathcal{C}^{\otimes})) = N(\mathrm{CAlg}(\mathcal{C}))$

Def [Groth 4.31]. $p: \mathcal{M}^{\otimes} \to N(\mathrm{Fin}_*)$

$q: \mathcal{N}^{\otimes} \to N(\mathrm{Fin}_*)$

symmetric monoidal ∞-categories.

$F: \mathcal{M}^{\otimes} \to \mathcal{N}^{\otimes}$ over $N(\mathrm{Fin}_*)$ is

(1) symmetric monoidal if it sends p-cocartesian edges to q-cocartesian edges.

$\mathrm{Fun}^{\otimes}(\mathcal{M}^{\otimes}, \mathcal{N}^{\otimes})$

(2) lax symmetric monoidal if it sends inert edges to inert edges.

edges to inert edges.

$\mathrm{Fun}^{\otimes, lax}(M^{\otimes}, N^{\otimes})$

$\mathrm{Fun}^{\otimes, lax}(N\mathrm{Fin}_*), N^{\otimes}) = \mathrm{CAlg}(N)$.

monoidal categories.

Def. A monoidal ∞-category as a coCartesian fibration of ∞-operads $\mathcal{C}^{\otimes} \to \mathrm{Ass}^{\otimes}$.

Prop (HA 4.1.1.14) $p: \mathcal{C}^{\otimes} \to N(\mathrm{Fin}_*)$ ∞-operad. then p is a symmetric monoidal ∞-cat $\Leftrightarrow$ the pullback $p': \mathcal{C}^{\otimes} \times_{N(\mathrm{Fin}_*)} \mathrm{Ass}^{\otimes} \to \mathrm{Ass}^{\otimes}$ is a monoidal ∞-cat.

Def. $\mathcal{C}^{\otimes}$: ∞-operad with fibration $q: \mathcal{C}^{\otimes} \to \mathrm{Ass}^{\otimes}$.

$$\mathrm{Alg}(\mathcal{C}) = \mathrm{Alg}_{/\mathrm{Ass}^{\otimes}}(\mathcal{C})$$

Another definition:

Def. A (A_∞-)monoidal ∞-category is a coCartesian fibration $p: M^{\otimes} \to N(\Delta^{op})$ such that the Segal maps are equivalences.

Rem. ρ^i is in image of $\Delta^{op} \to \mathrm{Fin}_*$

Def. A morphism $\alpha: [n] \to [k]$ in Δ is convex

Def. A morphism $\alpha : [n] \to [k]$ in Δ is convex if it's injective and its image is the interval $[\alpha[0], \alpha[n]]$.

rem: α is convex $\Leftrightarrow$ it's image in Fin_* is inert

Def. $p : M^{\otimes} \to N(\Delta^{op})$ $(A_\infty-)$ monoidal ∞-cat.

An $(A_\infty-)$ algebra object in $M^{\otimes}$ is a section $N : N(\Delta^{op}) \to M^{\otimes}$ such that convex morphisms are sent to p-cocartesian morphisms.

Def. $p : M^{\otimes} \to N(\Delta^{op})$ $\quad q : N^{\otimes} \to N(\Delta^{op})$ $(A_\infty-)$monoidal ∞-cats. $F : M^{\otimes} \to N^{\otimes}$ over $N(\Delta^{op})$ is (1) lax monoidal if it sends p-cocartesian lifts of convex morphisms to q-cocartesian morphisms

(2) monoidal if it sends p-coCart to q-coCart.

$$\mathrm{Alg}_{A_\infty}(M^{\otimes}) = \mathrm{Fun}^{\otimes, lax}(N(\Delta^{op}), M^{\otimes})$$

Rem: $p : M^{\otimes} \to N(\Delta^{op})$ is a planar ∞-operad, not an ∞-operad

Relation of the two definitions

The map $\Delta^{op} \to \mathrm{Fin}_*$ factors $\Delta^{op} \xhookrightarrow{\mathrm{Cut}} \mathrm{Ass}^{\otimes}$

Fin_*

Prop (HA 4.1.2.11) $\mathrm{Cut}: N(\Delta^{op}) \to \mathrm{Ass}^{\otimes}$ is an approximation to the ∞-operad $\mathrm{Ass}^{\otimes}$ (in the sense of HA 2.3.3.6.)

Prop (HA 4.1.3.19) $q: \mathcal{O}^{\otimes} \to \mathrm{Ass}^{\otimes}$ fibration of ∞-operads.
Then $\mathrm{Cut}: N(\Delta^{op}) \to \mathrm{Ass}^{\otimes}$ induces an equivalence of ∞-categories $\mathrm{Alg}(\mathcal{O}) \to \mathrm{Alg}_{A_\infty}(\mathcal{O})$.

§6. Modules.

fibration of ∞-operads $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$

$\Rightarrow$ ∞-category $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$

fix $A \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$

Assume $\mathcal{O}$ is unital (c.f. $\mathcal{O}(0) \simeq *$)

$\Rightarrow$ ∞-category $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})$ A-module objects in $\mathcal{C}$

Assume $\mathcal{O}$ is coherent (examples. $\mathrm{Comm}^{\otimes}$, $E_k^{\otimes}$)

$\Rightarrow$ fibrations of ∞-operads $p: \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathcal{O}^{\otimes}$.

In good cases,

$\Rightarrow$ p is a cocartesian fibration of ∞-operads

§7. structured ring spectra.

$E_0^{\otimes} \hookrightarrow E_1^{\otimes} \hookrightarrow \cdots \hookrightarrow E_\infty^{\otimes} = \mathrm{Comm}^{\otimes}$

$\mathrm{CAlg} \to \cdots \to \mathrm{Alg}_{E_2^{\otimes}}(Sp) \to \mathrm{Alg}_{E_1^{\otimes}}(Sp) \to \mathrm{Alg}_{E_0^{\otimes}}(Sp)$

$\simeq$ 4.8.2.22

monads T on Sp which preserves small colims

(HA 7.1.1)

$R \in \mathrm{Alg}_{E_1}(Sp) \Rightarrow \quad \mathrm{LMod}_R(Sp) =: \mathrm{LMod}_R$ ← defined in HA 4.2.1 using means of multicategories

$\mathrm{RMod}_R(Sp)$

- If $A \in \mathrm{Alg}_{E_\infty}(Sp)$, $\mathrm{Mod}_A^{E_\infty}(Sp) \xrightarrow{\sim} \mathrm{LMod}_A(Sp)$

- LMod_R – is a stable ∞-category.

(R connective) → – inherits an accessible t-structure.

– $\pi_0 : \mathrm{LMod}_R^{\heartsuit} \xrightarrow{\sim} N(\mathrm{Mod}_{\pi_0(R)})$

R : associative ring $\qquad \mathcal{A}$: abelian category of left R-mod

$\mathrm{LMod}_R^{\heartsuit} \simeq N(\mathcal{A}) \simeq \mathcal{D}^{-}(\mathcal{A})^{\heartsuit}$

$\Rightarrow \exists\ \theta : \mathcal{D}^{-}(\mathcal{A}) \longrightarrow \mathrm{LMod}_R$ right t-exact

fully-faithful, essential image = right bounded objects.

fully-faithful, essential image = right bounded objects.

$\Rightarrow \quad \mathcal{D}(\mathcal{A}) \xrightarrow{\sim} LMod_R$ both right complete & f.f.

$\mathcal{D}^-(\mathcal{A}) = N_{dg}(Ch^-(\mathcal{A}_{proj}))$ can be obtained from inverting q.i

$\mathcal{D}(\mathcal{A}) = N_{dg}(Ch(\mathcal{A})^\circ)$ $\mathcal{A}$: Grothendieck abelian cat.

$\exists F : \mathcal{D}^-(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ f.f. embedding, image $\bigcup_n \mathcal{D}(\mathcal{A})_{\geq -n}$

- R: comm ring, then $\sim$ of symmetric monoidal ∞-categories $\mathcal{D}(\mathcal{A}) \simeq Mod_R$.

Proof uses the following recognition principle:

Thm (Schwede-Shipley)

$\mathcal{C}$: stable ∞-cat. $\mathcal{C} \simeq RMod_R$ for some $R \in Alg_{E_1}$

$\Leftrightarrow$ (1) $\mathcal{C}$ is presentable

(2) there exists a compact object $C \in \mathcal{C}$ which generates $\mathcal{C}$ (in the sense that $Ext^n_{\mathcal{C}}(C, D) = 0$ for all $n \Rightarrow D \simeq 0$).

Rem. can take $R = End_{\mathcal{C}}(C)$

Thm. $1 \leq k \leq \infty$. $R \mapsto RMod_R^{\otimes}$ determines a fully-faithful embedding $Alg_{E_k} \to Alg_{E_{k-1}}(Pr^L)$.

a fully-faithful embedding $\mathrm{Alg}_{\mathbb{E}_k} \to \mathrm{Alg}_{\mathbb{E}_{k-1}}(\mathrm{Pr}^L)$.

Essential image: $\mathcal{C}^{\otimes} \to \mathbb{E}_{k-1}^{\otimes}$ such that

$\mathbb{E}_{k-1}$ monoidal presentable ∞-cats.

(1) $\mathcal{C}$ is stable and presentable.

(when $k>1$) $\otimes$ preserves small colimit in each variable

(2) The unit object $1 \in \mathcal{C}$ is compact.

(3) 1 generates $\mathcal{C}$.

— end —