Intro to ∞-category theory

Q: Why ∞-category thy?

A: Higher cat'l structures arise in algebra, geometry, math. pysics, number thy, etc.

Naive model: Take the (Quillen model) category of simplicially enriched categories, Cat_Δ.

(simplicial sets $sSet_\Delta = Fun(\Delta^{op}, set)$).

$\mathcal{C} \in Cat_\Delta$,

Def f $Map_{\mathcal{C}}(s,t) \xrightarrow{\sim} Map_{\mathcal{C}}(f(s), f(t))$.

$\forall s,t \in \mathcal{C}$.

And $\forall$ $t \in \mathcal{D}$, $\exists$ $s \in \mathcal{C}$ and an equivalence $f(s) \xrightarrow{\simeq} t$.

Not a good exponential in Cat_Δ.

Loosen this up. How?

$$\begin{tikzcd} Cat \arrow[r, "ft"] \arrow[d] & sSet_\Delta \arrow[dl, bend left, "\mathfrak{C}[-]"] \\ Cat_\Delta \arrow[ur, "N"] & \end{tikzcd}$$

$(N\mathcal{C})_n = Fun([n], \mathcal{C})$, $[n] = \{0 \to 1 \to \dots \to n\}$.

$\Delta \to Cat$

$$\downarrow h$$
$$\mathrm{Cat}_\Delta$$

$$\Delta^n \longmapsto \left\{ \begin{tikzcd} 0 \arrow[r] \arrow[rr, bend left, ""{name=A}] \arrow[rrr, bend left=50] & 1 \arrow[r] \arrow[to=A, phantom, "\Downarrow"] & 2 \arrow[r] & 3 \arrow[r] & \cdots \arrow[r] & n \end{tikzcd} \right\}$$

$$\begin{tikzcd} 1 \arrow[rr, bend right, "\Uparrow"] & 2 & 3 \end{tikzcd}$$

I.e. • $\mathrm{Map}(0,2) = \left(\begin{array}{c} 02 \\ \downarrow \\ 012 \end{array} \right) \cong \Delta^1$

• $\mathrm{Map}(0,3) = \left(\begin{array}{ccc} 03 & \longrightarrow & 023 \\ \downarrow & & \downarrow \\ 013 & \longrightarrow & 0123 \end{array} \right) \cong \Delta^1 \times \Delta^1$

etc.,

$$\mathrm{Map}(i,j) \cong \begin{cases} \emptyset & j<i \\ \Delta^0 & j=i \\ (\Delta^1)^{j-i-1} & j>i \end{cases}.$$

<u>Thm:</u> (Lurie) $N: \mathrm{Cat}_\Delta \longrightarrow \mathrm{Set}_\Delta$ is right Quillen functor from the Bergner m.s. on Cat_Δ to Joyal m.s. on Set_Δ.

- fibrant objects are quasicategories.
- weak equivs are ff and ees. as before.

In Set_Δ, if $\mathcal{C}$ and $\mathcal{D}$ are ∞-cats (Joyal fibrant) then $\mathrm{Fun}(\mathcal{C},\mathcal{D}) = \mathcal{D}^{\mathcal{C}}$ is homotopically well-behaved.

($\star$ = join)

<u>Join</u>: Define a monoidal product on Set_Δ by $\Delta^m \star \Delta^n = \Delta^{m+1+n}$.

$$\{0 \to \cdots \to m\} \star \{0 \to \cdots \to n\} = \{0 \to 1 \to \cdots \to m \to m+1 \to \cdots \to m+1+n\}$$

Extend to Set_Δ by colims:

$$X \star Y = \operatorname*{colim}_{\Delta^m \to X,\ \Delta^n \to Y} \Delta^m \star \Delta^n .$$

If $f : \mathcal{D} \to \mathcal{C}$ is a functor, then $\mathcal{C}_{/f}$ (the slice of $\mathcal{C}$ over $f : \mathcal{D} \to \mathcal{C}$) has n-simplices $(\mathcal{C}_{/f})_n = \{\Delta^n \star \mathcal{D} \to \mathcal{C}\}$

$$\emptyset \star \mathcal{D} \cong \mathcal{D} \xrightarrow{f} \mathcal{C} .$$

In particular, if $\mathcal{D} = *$, $f \in \mathcal{C}$.

$\mathcal{C}_{/f}$ is just the usual "slice of $\mathcal{C}$ over the object f".

$$\{\Delta^n \to \mathcal{C}_{/f}\} = \{\Delta^{n+1} \to \mathcal{C} \mid \text{final vertex is } f\}.$$

<u>Defn</u>: If $\mathcal{C}$ is an ∞-cat, and $s, t \in \mathcal{C}$, then $\mathrm{Map}_{\mathcal{C}}(s,t) \xrightarrow{fib} \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\partial\Delta^1, \mathcal{C})$ is fiber over $(s,t) \in \mathcal{C} \times \mathcal{C}$.

Def'n: An object $t \in \mathcal{C}$ is terminal if $\forall\, s \in \mathcal{C}$, $\mathrm{Map}_{\mathcal{C}}(s,t) \xrightarrow{\sim} *$.

Def'n: Let $f : \mathcal{D} \to \mathcal{C}$ be a functor of ∞-cats. Then a limit of f is a terminal object of $\mathcal{C}_{/f}$.

Eg: If $\mathcal{C}$ is a fibrant simplicial category, then homotopy (co)limits in $\mathcal{C}$ compute (co)limits in $N\mathcal{C}$.

Fibrations

If $S \in \mathrm{Set}$, $\mathrm{Set}_{/S} \xleftarrow{\sim} \mathrm{Fun}(S^{op}, \mathrm{Set})$

Grothendieck construction $X = \int f \longleftarrow\!\shortmid f \searrow *$

$$\int f = \{(s \in S, x \in f(s))\}.$$

$$\begin{array}{ccc} X & \xrightarrow{p} & S \\ \| & & \| \\ \int f & & \int * \end{array}$$

$\mathrm{Cat}_\infty := N(\mathrm{QCat})$

$\mathrm{Gpd}_\infty := N(\mathrm{Kan})$

the simp. set of maps $\mathcal{C}$ to $\mathcal{D}$ is max'l Kan complex inside $\mathrm{Fun}(\mathcal{C},\mathcal{D})$.

$\mathrm{QCat} \subseteq \mathrm{Set}_\Delta$

$\mathrm{Kan} \subseteq \mathrm{Set}_\Delta$

Thm: (Classifying spaces)

$$\mathrm{Cat}_{\infty/S} \supset \left(\mathrm{Cat}_{\infty/S}\right)^{\text{right}} \underset{\simeq}{=} \mathrm{Gpd}_{\infty/S} \xleftarrow[\int]{\sim} \mathrm{Fun}(S^{op}, \mathrm{Gpd}_\infty)$$

$$\{(s \in S, x \in f(s))\} \longleftarrow\!\shortmid \; f$$

"right fibrations"

"Cartesian fibrations".

This fails if S is not an ∞-gpd:

Thm: (Lurie)

$$\left(\mathrm{Cat}_{\infty/S}\right)^{\text{cart}} \xleftarrow[\int]{\sim} \mathrm{Fun}(S^{op}, \mathrm{Cat}_\infty)$$

$$\int f \longleftarrow\!\shortmid \; f.$$

Moral: $p : X \to S$ is a cartesian fibration

(contravariantly)

$\iff$ $s \longmapsto X_s$ is functorial in S.

Def'n: A map $p : X \to S$ in Cat_∞ is a cartesian fibration if, for every map $f : s \to t \in S$ and every object $y \in X$, there exists a cartesian lift of f to y: i.e. a map $g : x \to y \in X$ st $p(g) = f$ and $X_{/g} \xrightarrow{\sim} X_{/y} \times_{S_{/t}} S_{/f}$.

E.g. $\mathrm{Vect} \to \mathrm{Man}$, a cartesian fibration because vector bundles pull back.

There's a universal cocartesian fibration

$$\mathrm{Cat}_{\infty *//} \simeq \int \mathrm{id}_{\mathrm{Cat}_\infty} \longrightarrow \int *_{\mathrm{Cat}_\infty} \simeq \mathrm{Cat}_\infty$$

Because of this and $\mathrm{Cat}_{\infty/S}^{\mathrm{cocart}} \simeq \mathrm{Fun}(S, \mathrm{Cat}_\infty)$ any cocartesian fibration pulls back:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "p"'] & \mathrm{Cat}_{\infty *//} \arrow[d] \\ S \arrow[r, "f"] & \mathrm{Cat}_\infty \end{tikzcd}$$

Defn: An adjunction in Cat_∞ is a map $Z \to \Delta^1$ which is cartesian and cocartesian.

$$\begin{tikzcd} \mathcal{C} = Z_0 \arrow[r, bend left, "f"] & \mathcal{D} = Z_1 \arrow[l, bend left, "g"] \end{tikzcd}$$

Want to define a subcategory of CAT_∞ the presentable ∞-cats.

Thm: (Lurie) If S is an ∞-cat, then $\mathcal{P}(S) = \mathrm{Fun}(S^{op}, \mathrm{Gpd}_\infty)$ is the universal cocompletion of S.

I.e. if T cocomplete, then

$$\mathrm{Fun}^{\mathrm{colim}}(\mathcal{P}(S), T) \xrightarrow{\sim} \mathrm{Fun}(S, T).$$

restriction along Yoneda.

Defn: An ∞-cat $\mathcal{C}$ is presentable if there's a small ∞-cat S and a left adjoint $\mathcal{P}(S) \to \mathcal{C}$ ("small gens") st

- the right adjoint $\mathcal{C} \xrightarrow{R} \mathcal{P}(S)$ is fully faithful and R preserves κ-filtered colims for some $\kappa \gg 0$.

("small relations")

Thm: (Adjoint functor thm)

A functor of pres. ∞-cats $f: \mathcal{C} \to \mathcal{D}$ is a left adjoint iff f preserves colims.