Chain-level models of equivariant topological Hochschild homology in positive characteristic

Po Hu

Wayne State University

March 7, 2025

Outline

- Joint with V. Burghardt, I. Kriz, P. Somberg
- For an $\mathbb{F}_p$-algebra R, its $\mathbb{Z}/p^r$-equivariant topological Hochschild homology $THH_{\mathbb{Z}/p^r}(R)$ is a module over the E_∞-algebra $TR(\mathbb{F}_p) = H\underline{\mathbb{Z}_p}$, where $\underline{\mathbb{Z}_p}$ is the constant $\mathbb{Z}/p^r$-Mackey functor for the ring $\mathbb{Z}_p$
- There is an equivalence between the derived category of $H\underline{\mathbb{Z}_p}$-modules and the derived category of $\underline{\mathbb{Z}_p}$-module Mackey functors

Outline

- Joint with V. Burghardt, I. Kriz, P. Somberg
- For an $\mathbb{F}_p$-algebra R, its $\mathbb{Z}/p^r$-equivariant topological Hochschild homology $THH_{\mathbb{Z}/p^r}(R)$ is a module over the E_∞-algebra $TR(\mathbb{F}_p) = H\underline{\mathbb{Z}_p}$, where $\underline{\mathbb{Z}_p}$ is the constant $\mathbb{Z}/p^r$-Mackey functor for the ring $\mathbb{Z}_p$
- There is an equivalence between the derived category of $H\underline{\mathbb{Z}_p}$-modules and the derived category of $\underline{\mathbb{Z}_p}$-module Mackey functors

This talk: For a quasi-smooth semiperfect $\mathbb{F}_p$-algebra R, give an explicit description of $THH_{\mathbb{Z}/p^r}(R)$ as a chain complex of $\underline{\mathbb{Z}_p}$-modules. By semiperfect descent, we can thus also do this for R smooth over $\mathbb{F}_p$.

Classical Hochschild homology

For a ring R, the *cyclic bar construction* of R has

$$B_n^{cyc} = R^{\otimes(n+1)}$$

with face maps

$$d_i(a_0 \otimes \cdots a_n) = \begin{cases} a_0 \otimes \cdots a_i a_{i+1} \otimes a_n & 0 \leq i \leq n-1 \\ a_n a_0 \otimes \cdots a_{n-1} & i = n \end{cases}$$

Form the associated chain complex $C_\bullet(R)$, the *Hochschild homology* of R is

$$HH_*(R) = H_*(C_\bullet(R)) = \pi_*|B_\bullet^{cyc}(R)|.$$

Topological Hochschild homology

Do this in a symmetric monoidal category of spectra, e. g. S-modules. Let R be associative S-algebra.

Bökstedt, EKMM: cyclic bar construction with $B_n^{cyc}(R) = R^{\wedge_S n}$ gives

$$THH(R) = |B_n^{cyc}(R)|.$$

Loday construction: for R commutative, there is a simplicial set model for S^1 with $n+1$ n-simplices,

$$THH(R) = |R \otimes S^1|.$$

Topological Hochschild homology

Do this in a symmetric monoidal category of spectra, e. g. S-modules. Let R be associative S-algebra.

Bökstedt, EKMM: cyclic bar construction with $B_n^{cyc}(R) = R^{\wedge_S n}$ gives

$$THH(R) = |B_n^{cyc}(R)|.$$

Loday construction: for R commutative, there is a simplicial set model for S^1 with $n+1$ n-simplices,

$$THH(R) = |R \otimes S^1|.$$

- One can put the structure of a genuine S^1-equivariant spectrum on $THH(R)$. In practice, work with finite, i. e. cyclic subgroups.
- We work with the genuine $\mathbb{Z}/p^r$-equivariant spectrum $THH_{\mathbb{Z}/p^r}(R)$.

The cyclotomic property

Recall the isotropy separation sequence

$$E\mathbb{Z}/p^r_+ \to S^0 \to \widetilde{E\mathbb{Z}/p^r}$$

The $\mathbb{Z}/p$-geometric fixed points of $THH_{\mathbb{Z}/p^r}(R)$ is

$$\Phi^{\mathbb{Z}/p} THH_{\mathbb{Z}/p^r}(R) = (\widetilde{E\mathbb{Z}/p^r} \wedge THH_{\mathbb{Z}/p^r}(R))^{\mathbb{Z}/p}.$$

The cyclotomic property

Recall the isotropy separation sequence

$$E\mathbb{Z}/p^r_+ \to S^0 \to \widetilde{E\mathbb{Z}/p^r}$$

The $\mathbb{Z}/p$-geometric fixed points of $THH_{\mathbb{Z}/p^r}(R)$ is

$$\Phi^{\mathbb{Z}/p} THH_{\mathbb{Z}/p^r}(R) = (\widetilde{E\mathbb{Z}/p^r} \wedge THH_{\mathbb{Z}/p^r}(R))^{\mathbb{Z}/p}.$$

The cyclotomic property ($\mathbb{Z}/p^r$-equivariant):

$$\Phi^{\mathbb{Z}/p} THH_{\mathbb{Z}/p^{r+1}}(R) \simeq THH_{\mathbb{Z}/p^r}(R).$$

So we get a map of $\mathbb{Z}/p^r$-spectra

$$R : THH_{\mathbb{Z}/p^{r+1}}(R)^{\mathbb{Z}/p} \to THH_{\mathbb{Z}/p^r}(R)$$

$\mathbb{Z}/p^r$-equivariant topological restriction homology

Iterating the map R, define the $\mathbb{Z}/p^r$-equivariant spectrum $TR_{\mathbb{Z}/p^r}(R)$ as

$$TR_{\mathbb{Z}/p^r}(R) = \text{holim}\ (\cdots \to THH_{\mathbb{Z}/p^{r+1}}(R)^{\mathbb{Z}/p} \to THH_{\mathbb{Z}/p^r}(R))$$

$\mathbb{Z}/p^r$-equivariant topological restriction homology

Iterating the map R, define the $\mathbb{Z}/p^r$-equivariant spectrum $TR_{\mathbb{Z}/p^r}(R)$ as

$$TR_{\mathbb{Z}/p^r}(R) = \text{holim } (\cdots \to THH_{\mathbb{Z}/p^{r+1}}(R)^{\mathbb{Z}/p} \to THH_{\mathbb{Z}/p^r}(R))$$

We will work with $\mathbb{F}_p$-algebras R. They have the Frobenius map $\varphi : R \to R$, namely $\varphi(x) = x^p$. R is

- *perfect* if φ is an isomorphism
- *semiperfect* if φ is surjective

Witt vectors

For a commutative ring R, recall the ring of (p-typical) *Witt vectors*. As a set,

$$W(R) = \{(a_0, a_1, \dots) \mid a_i \in R\}.$$

with

$$(a_0, a_1, \dots) + (b_0, b_1, \dots) = (s_0(a_0, b_0), s_1(a_0, a_1, b_0, b_1), \dots)$$
$$(a_0, a_1, \dots) \cdot (b_0, b_1, \dots) = (p_0(a_0, b_0), p_1(a_0, a_1, b_0, b_1), \dots)$$

The integral polynomials $s_0, s_1, \dots, p_0, p_1, \dots$ are determined by the requirement that the ghost map $w : W(\mathbb{Z}) \to \mathbb{Z}^{\mathbb{N}_0}$ given by

$$(a_0, a_1, a_2, \dots) \mapsto (a_0, a_0^p + pa_1, a_0^{p^2} + pa_1^p + p^2 a_2, \dots)$$

be a ring map.

On the ring of Witt vectors $W(R)$, there is the *Frobenius* ring map and the additive *Verschiebung* map

$$F(w_0, w_1, \ldots) = (w_1, w_2, \ldots)$$
$$V(a_0, a_1, \ldots) = (0, a_0, a_1, \ldots)$$

They satisify

- $xV(y) = V(F(x)y)$ and $FV = p$
- For a $\mathbb{F}_p$-algebra R, $VF = p$
- For a $\mathbb{F}_p$-algebra R, $F = W(\varphi)$: $F(a_0, a_1, \ldots) = (a_0^p, a_1^p, \ldots)$
- Length n Witt vectors: $W_n(R) = W(R)/V^n W(R)$

Example: $W(\mathbb{F}_p) = \mathbb{Z}_p$, $W_k(\mathbb{F}_p) = \mathbb{Z}/p^k$.

The perfect case

Theorem (Hesselholt-Mandell)

For any commutative ring R,

$$THH(R)_0^{\mathbb{Z}/p^k} = W_{k+1}(R).$$

Theorem (Hesselholt-Madsen)

For R a perfect $\mathbb{F}_p$-algebra,

$$THH(R)_*^{\mathbb{Z}/p^k} = W_{k+1}(R)[\sigma_k]$$

where $|\sigma_k| = 2$.

The map

$$R : THH(R)^{\mathbb{Z}/p^k} \to THH(R)^{\mathbb{Z}/p^{k-1}}$$

sends σ_k to $p\sigma_{k-1}$. For $R = \mathbb{F}_p$ and interpreting the result in the $\mathbb{Z}/p^r$-equivariant category, passing to inverse limit gives

$$TR_{\mathbb{Z}/p^r}(\mathbb{F}_p) \simeq H\underline{\mathbb{Z}_p}.$$

The map

$$R : THH(R)^{\mathbb{Z}/p^k} \to THH(R)^{\mathbb{Z}/p^{k-1}}$$

sends σ_k to $p\sigma_{k-1}$. For $R = \mathbb{F}_p$ and interpreting the result in the $\mathbb{Z}/p^r$-equivariant category, passing to inverse limit gives

$$TR_{\mathbb{Z}/p^r}(\mathbb{F}_p) \simeq H\underline{\mathbb{Z}_p}.$$

Leads to: For $\mathbb{F}_p$-algebra R, $THH_{\mathbb{Z}/p^r}(R)$ is a module over the E_∞-algebra $TR_{\mathbb{Z}/p^r}(\mathbb{F}_p) = H\underline{\mathbb{Z}_p}$.

Proof: For simplicity, just take $R = \mathbb{F}_p$. Do induction on r. One has the Tate diagram (of nonequivariant spectra) for cyclotomic spectra. Here, the right half is

$$\begin{array}{ccc}
THH(\mathbb{F}_p)^{\mathbb{Z}/p^k} & \xrightarrow{\quad R \quad} & \overbrace{(\Phi^{\mathbb{Z}/p} THH(\mathbb{F}_p))^{\mathbb{Z}/p^{k-1}}}^{THH(\mathbb{F}_p)^{\mathbb{Z}/p^{k-1}}} \\
\downarrow & & \downarrow \\
\underbrace{F(E\mathbb{Z}/p^k_+, THH(\mathbb{F}_p))^{\mathbb{Z}/p^k}}_{THH(\mathbb{F}_p)^{h\mathbb{Z}/p^k}} & \longrightarrow & \underbrace{(\widetilde{E\mathbb{Z}/p^k} \wedge F(E\mathbb{Z}/p^k_+, THH(\mathbb{F}_p)))^{\mathbb{Z}/p^k}}_{THH(\mathbb{F}_p)^{t\mathbb{Z}/p^k}}
\end{array}$$

Proof: For simplicity, just take $R = \mathbb{F}_p$. Do induction on r. One has the Tate diagram (of nonequivariant spectra) for cyclotomic spectra. Here, the right half is

$$\begin{array}{ccc}
THH(\mathbb{F}_p)^{\mathbb{Z}/p^k} & \xrightarrow{R} & \overbrace{(\Phi^{\mathbb{Z}/p} THH(\mathbb{F}_p))^{\mathbb{Z}/p^{k-1}}}^{THH(\mathbb{F}_p)^{\mathbb{Z}/p^{k-1}}} \\
\downarrow & & \downarrow \\
\underbrace{F(E\mathbb{Z}/p^k_+, THH(\mathbb{F}_p))^{\mathbb{Z}/p^k}}_{THH(\mathbb{F}_p)^{h\mathbb{Z}/p^k}} & \longrightarrow & \underbrace{(\widetilde{E\mathbb{Z}/p^k} \wedge F(E\mathbb{Z}/p^k_+, THH(\mathbb{F}_p)))^{\mathbb{Z}/p^k}}_{THH(\mathbb{F}_p)^{t\mathbb{Z}/p^k}}
\end{array}$$

Top right: by induction hypothesis,

$$THH(\mathbb{F}_p)_*^{\mathbb{Z}/p^{k-1}} = \mathbb{Z}/p^k[\sigma_{k-1}].$$

Bottom row: The Tate spectral sequence has

$$E_2 = \mathbb{Z}/p[t, t^{-1}, \sigma] \otimes \Lambda_{\mathbb{Z}/p}[u]$$

with $|t| = (-2, 0), |\sigma| = (0, 2), |u| = (-1, 0)$. The only differentials are

$$d^{2k+1} : u \mapsto t^{k+1}\sigma^k.$$

Together with extensions, this gives

$$THH(\mathbb{F}_p)_*^{t\mathbb{Z}/p^k} = \mathbb{Z}/p^k[\sigma_{k-1}^{\pm 1}]$$

and

$$THH(\mathbb{F}_p)_*^{h\mathbb{Z}/p^k} = \mathbb{Z}/p^{k+1}[\sigma_k] \oplus \sigma_{k-1}^{-1} \cdot \mathbb{Z}/p^k[\sigma_{k-1}^{-1}]$$

with $\sigma_k \to p\sigma_{k-1}$. This gives the upper left corner.

Starting the induction

Theorem (Bökstedt, Breer)

$$THH_*(\mathbb{F}_p) = \mathbb{F}_p[\sigma]$$

where $|\sigma| = 2$.

Starting the induction

Theorem (Bökstedt, Breer)

$$THH_*(\mathbb{F}_p) = \mathbb{F}_p[\sigma]$$

where $|\sigma| = 2$.

Bökstedt: skeleton spectral sequence

$$HH_*(A_*) = Tor^{A_* \otimes A_*}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow H_* THH(\mathbb{F}_p)$$

Starting the induction

Theorem (Bökstedt, Breer)

$$THH_*(\mathbb{F}_p) = \mathbb{F}_p[\sigma]$$

where $|\sigma| = 2$.

Bökstedt: skeleton spectral sequence

$$HH_*(A_*) = Tor^{A_* \otimes A_*}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow H_* THH(\mathbb{F}_p)$$

Blumberg-Cohen-Schlichtkrull: considering THH of generalized Thom spectra shows that

$$THH(\mathbb{F}_p) \simeq H\mathbb{F}_p \wedge \Omega S^3_+.$$

A theorem of Hopkins-Mahowald leads to the right hand side.

A slightly different variant of Bökstedt's proof:

$$
\begin{aligned}
THH(\mathbb{F}_p) &= \left|B_S(H\mathbb{F}_p, H\mathbb{F}_p \wedge_S H\mathbb{F}_p, H\mathbb{F}_p)\right| \\
&= \left|H\mathbb{F}_p \wedge_{H\mathbb{F}_p \wedge_S H\mathbb{F}_p} B_S(H\mathbb{F}_p \wedge_S H\mathbb{F}_p, H\mathbb{F}_p \wedge_S H\mathbb{F}_p, H\mathbb{F}_p)\right| \\
&= \left|B_{H\mathbb{F}_p}(H\mathbb{F}_p, H\mathbb{F}_p \wedge_S H\mathbb{F}_p, H\mathbb{F}_p)\right|
\end{aligned}
$$

giving a spectral sequence

$$
Tor^{A_*}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow THH(\mathbb{F}_p)_*
$$

Bökstedt: the spectral sequence collapses for $p = 2$, "Kudo" differential d^{p-1} for $p > 2$.

The Mackey functor point of view

For a finite group G, *G-Mackey functors* are coefficient systems for genuine G-equivariant Eilenberg-Maclane spectra.

- The category of G-Mackey functors has a symmetric monoidal structure by the *box product*
- A *Green functor* is an algebra in G-Mackey functors with respect to the box product
- Thus, we can consider modules over a Green functor

The Mackey functor point of view

For a finite group G, *G-Mackey functors* are coefficient systems for genuine G-equivariant Eilenberg-Maclane spectra.

- The category of G-Mackey functors has a symmetric monoidal structure by the *box product*
- A *Green functor* is an algebra in G-Mackey functors with respect to the box product
- Thus, we can consider modules over a Green functor

Theorem (Mandell)

There is an equivalence between the derived category of modules over the E_∞-algebra $H\underline{\mathbb{Z}_p}$ and the derived category of modules over the Green functor $\underline{\mathbb{Z}_p}$.

Examples: $G = \mathbb{Z}/p^r,\ k < r$

$$
\begin{array}{ccccc}
\mathbb{Z}/p^r & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & W_{r+1}(R) \\
 & 1\downarrow\uparrow p & \downarrow\uparrow & \downarrow\uparrow & F\downarrow\uparrow V \\
\mathbb{Z}/p^{r-1} & \mathbb{Z} & \mathbb{Z}[\mathbb{Z}/p] & \mathbb{Z}[\mathbb{Z}/p] & W_r(R) \\
 & 1\downarrow\uparrow p & \downarrow\uparrow & \downarrow\uparrow & F\downarrow\uparrow V \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
 & 1\downarrow\uparrow p & \downarrow\uparrow & \downarrow\uparrow & F\downarrow\uparrow V \\
\mathbb{Z}/p^{r-k} & \mathbb{Z} & \mathbb{Z}[\mathbb{Z}/p^k] & \mathbb{Z}[\mathbb{Z}/p^k] & W_{r-k+1}(R) \\
 & 1\downarrow\uparrow p & \downarrow\uparrow & 1\downarrow\uparrow & \downarrow\uparrow \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
 & 1\downarrow\uparrow p & \downarrow\uparrow & 1\downarrow\uparrow & F\downarrow\uparrow V \\
\{e\} & \mathbb{Z} & \mathbb{Z}[\mathbb{Z}/p^r] & \mathbb{Z}[\mathbb{Z}/p^k] & R \\
 & & & & \\
\text{isotropy} & \underline{\mathbb{Z}} & \underline{\mathbb{Z}}[\mathbb{Z}/p^r] & \underline{\mathbb{Z}}[\mathbb{Z}/p^k] & \underline{W}_{r+1}(R)
\end{array}
$$

The perfect case: Mackey point of view

Choose $s > r$. Let α_s be the irreducible representation of $\mathbb{Z}/p^s$ where the generator acts by $e^{2\pi i/p^s}$. Define the chain complex of $\underline{\mathbb{Z}}$-modules

$$\mathcal{W}_{r,j} = \underline{\widetilde{C_*}}\left(S^{\infty\alpha_s^{p^{s-r-1-j}}}\right)^{\mathbb{Z}/p^{s-r}}$$

to be the $\mathbb{Z}/p^r$-Mackey functor valued reduced cellular chain complex.

Taking the subgroup $\mathbb{Z}/p^{s-r-j} \subset \mathbb{Z}/p^s$,

$$S^{\infty\alpha_s^{p^{s-r-1-j}}} = E\mathcal{F}[\widetilde{\mathbb{Z}/p^{s-r-j}}]$$

depends only on r, j.

Additively, the chain complex $\mathcal{W}_{r,0}$ looks like

$$\cdots \xrightarrow{1-\gamma} \underline{\mathbb{Z}}[\mathbb{Z}/p^r] \xrightarrow{pN} \underline{\mathbb{Z}}[\mathbb{Z}/p^r] \xrightarrow{1-\gamma} \underline{\mathbb{Z}}[\mathbb{Z}/p^r] \xrightarrow{p\epsilon} \underline{\mathbb{Z}}$$

where $\mathbb{Z}/p^r = \langle\gamma\rangle$, and $N = 1 + \gamma + \cdots + \gamma^{p^r-1}$.

The chain complex $\mathcal{W}_{r,j}$ for $j > 0$ looks like

$$\cdots \xrightarrow{1-\gamma} \underline{\mathbb{Z}}[\mathbb{Z}/p^{r-j+1}] \xrightarrow{N_{r-j+1}} \underline{\mathbb{Z}}[\mathbb{Z}/p^{r-j+1}] \xrightarrow{1-\gamma} \underline{\mathbb{Z}}[\mathbb{Z}/p^{r-j+1}] \xrightarrow{\epsilon} \underline{\mathbb{Z}}$$

where $N_k = 1 + \gamma + \cdots + \gamma^{p^k-1}$.

The homology of $\mathcal{W}_{r,j}$ are in nonnegative even dimensions.

$$
\begin{array}{ccccccc}
\mathbb{Z}/p^{r+1} & & \mathbb{Z}/p^{r} & & & & \mathbb{Z}/p \\
\downarrow\uparrow & & \downarrow\uparrow & & & & \downarrow\uparrow \\
\mathbb{Z}/p^{r} & & \mathbb{Z}/p^{r-1} & & & & 0 \\
\downarrow\uparrow & & \downarrow\uparrow & & & & \downarrow\uparrow \\
\vdots & & \vdots & & \cdots & & \vdots \\
\downarrow\uparrow & & \downarrow\uparrow & & & & \downarrow\uparrow \\
\mathbb{Z}/p^{2} & & \mathbb{Z}/p & & & & 0 \\
\downarrow\uparrow & & \downarrow\uparrow & & & & \downarrow\uparrow \\
\mathbb{Z}/p & & 0 & & & & 0 \\
 & & & & & & \\
H_{2q}(\mathcal{W}_{r,0}) & & H_{2q}(\mathcal{W}_{r,1}) & & \cdots & & H_{2q}(\mathcal{W}_{r,r})
\end{array}
$$

Two caveats:

1. The complexes $\mathcal{W}_{r,j}$ are E_∞-algebras in the derived category of $\underline{\mathbb{Z}}$-modules. But they are not in general strictly commutative DGAs. (The corresponding Tate complexes have non-trivial Steenrod operations, which translate to non-trivial Dyer-Lashof operations on $\mathcal{W}_{r,j}$.)

Two caveats:

1. The complexes $\mathcal{W}_{r,j}$ are E_∞-algebras in the derived category of $\underline{\mathbb{Z}}$-modules. But they are not in general strictly commutative DGAs. (The corresponding Tate complexes have non-trivial Steenrod operations, which translate to non-trivial Dyer-Lashof operations on $\mathcal{W}_{r,j}$.)

2. Even though their homologies consist of $\underline{\mathbb{Z}/p^{r-j+1}}$-modules, the $\mathcal{W}_{r,j}$ are not in general equivalent to chain complexes of $\underline{\mathbb{Z}/p^{r-j+1}}$-modules.

Theorem (Burghardt, H., Kriz, Somberg)

For a perfect $\mathbb{F}_p$-algebra R, the $H\underline{\mathbb{Z}_p}$-algebra $THH_{\mathbb{Z}/p^r}(R)$ is

$$\mathcal{W}_{r,0} \otimes_{\mathbb{Z}} W(R)$$

in $D\underline{\mathbb{Z}_p}$-Modules.

Theorem (Burghardt, H., Kriz, Somberg)

For a perfect $\mathbb{F}_p$-algebra R, the $H\underline{\mathbb{Z}_p}$-algebra $THH_{\mathbb{Z}/p^r}(R)$ is

$$\mathcal{W}_{r,0} \otimes_{\mathbb{Z}} W(R)$$

in $D\underline{\mathbb{Z}_p}$-Modules.

Proof sketch: Follows from the theorem of Hesselholt-Madsen, using the Tate diagram and induction on r via the cyclotomic property. We have

$$\begin{aligned} THH_{\mathbb{Z}/p^r}(R) &\simeq \Phi^{\mathbb{Z}/p^{s-r}} TR_{\mathbb{Z}/p^s}(R) \\ &= (E\widetilde{\mathcal{F}[\mathbb{Z}/p^{s-r}]} \wedge TR_{\mathbb{Z}/p^s}(R))^{\mathbb{Z}/p^{s-r}} \end{aligned}$$

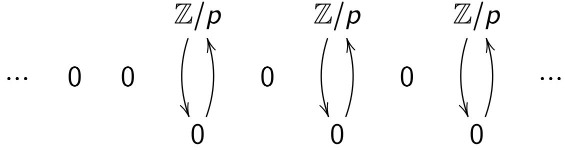
which gives rise to $\mathcal{W}_{r,0}$.

A digression

For $r = 1$, consider our model for $\mathcal{W}_{1,1} = \underline{\widetilde{C_*}}(S^{\infty\alpha})$:

$$\cdots \xrightarrow{1-\gamma} \underline{\mathbb{Z}}[\mathbb{Z}/p] \xrightarrow{N} \underline{\mathbb{Z}}[\mathbb{Z}/p] \xrightarrow{1-\gamma} \underline{\mathbb{Z}}[\mathbb{Z}/p] \xrightarrow{\epsilon} \underline{\mathbb{Z}}$$

This has homology $H_{2q}(\mathcal{W}_{1,1}) = \underline{\mathbb{Z}/p}_{\varphi}$ for $q \geq 0$ (and $H_{2q-1}(\mathcal{W}_{1,1}) = 0$):



A digression

For $r = 1$, consider our model for $\mathcal{W}_{1,1} = \underline{\widetilde{C_*}}(S^{\infty\alpha})$:

$$\cdots \xrightarrow{1-\gamma} \underline{\mathbb{Z}}[\mathbb{Z}/p] \xrightarrow{N} \underline{\mathbb{Z}}[\mathbb{Z}/p] \xrightarrow{1-\gamma} \underline{\mathbb{Z}}[\mathbb{Z}/p] \xrightarrow{\epsilon} \underline{\mathbb{Z}}$$

This has homology $H_{2q}(\mathcal{W}_{1,1}) = \underline{\mathbb{Z}/p}_{\varphi}$ for $q \geq 0$ (and $H_{2q-1}(\mathcal{W}_{1,1}) = 0$):

$$\begin{array}{ccccccccc} & & & \mathbb{Z}/p & & \mathbb{Z}/p & & \mathbb{Z}/p & \\ \cdots & 0 & 0 & \downarrow\uparrow & 0 & \downarrow\uparrow & 0 & \downarrow\uparrow & \cdots \\ & & & 0 & & 0 & & 0 & \end{array}$$

As a $\mathbb{Z}/p$-equivariant $H\underline{\mathbb{Z}}$-module, it corresponds to

$$S^{\infty\alpha} \wedge H\underline{\mathbb{Z}}.$$

Now consider the standard t-structure on $D(\underline{\mathbb{Z}}\text{-Modules})$: the heart is the category of $\underline{\mathbb{Z}}$-modules. We have the distinguished triangle

$$\tau_{\geq 2}\tau_{\leq 2}\underline{\widetilde{C_*}}(S^{\infty\alpha}) \to \tau_{\leq 2}\underline{\widetilde{C_*}}(S^{\infty\alpha}) \to \tau_{\leq 0}\underline{\widetilde{C_*}}(S^{\infty\alpha})$$

which is

$$\underline{\mathbb{Z}/p}_{\varphi}[2] \to X \to \underline{\mathbb{Z}/p}_{\varphi}$$

So

$$[X] \in Ext^3(\underline{\mathbb{Z}/p}_{\varphi}, \underline{\mathbb{Z}/p}_{\varphi}).$$

Now consider the standard t-structure on $D(\underline{\mathbb{Z}}\text{-Modules})$: the heart is the category of $\underline{\mathbb{Z}}$-modules. We have the distinguished triangle

$$\tau_{\geq 2}\tau_{\leq 2}\underline{\widetilde{C_*}}(S^{\infty\alpha}) \to \tau_{\leq 2}\underline{\widetilde{C_*}}(S^{\infty\alpha}) \to \tau_{\leq 0}\underline{\widetilde{C_*}}(S^{\infty\alpha})$$

which is

$$\underline{\mathbb{Z}/p}_{\varphi}[2] \to X \to \underline{\mathbb{Z}/p}_{\varphi}$$

So

$$[X] \in Ext^3(\underline{\mathbb{Z}/p}_{\varphi},\ \underline{\mathbb{Z}/p}_{\varphi}).$$

Turns out: $[X] \neq 0 \in Ext^3(\underline{\mathbb{Z}/p}_{\varphi},\ \underline{\mathbb{Z}/p}_{\varphi})$. (By writing down explicit resolutions in $Ext^3(\underline{\mathbb{Z}}_{\varphi}, \underline{\mathbb{Z}/p}_{\varphi}) = Hom(\underline{\mathbb{Z}}, \underline{\mathbb{Z}/p}_{\varphi})$.)

Compare $\underline{\widetilde{C_*}}(S^\infty)$ with the $\mathbb{Z}/p$-spectrum $S^{\infty\alpha} \wedge H\underline{\mathbb{Z}}$, which is also an $S^{\infty\alpha}$-module. The fixed points functor $(-)^{\mathbb{Z}/p}$ gives an equivalence $S^{\infty\alpha}$-Mod $\simeq$ Spectra, where

$$S^{\infty\alpha} \wedge H\underline{\mathbb{Z}} \longmapsto H\mathbb{Z}/p \vee \Sigma^2 H\mathbb{Z}/p \vee \Sigma^4 H\mathbb{Z}/p \vee \cdots .$$

So in the standard t-structure on $\mathbb{Z}/p$-equivariant spectra (with the heart being Mackey functors),

$$\tau_{\leq 2}(S^{\infty\alpha} \wedge H\underline{\mathbb{Z}}) = H\underline{\mathbb{Z}/p}_{\varphi} \vee H\underline{\mathbb{Z}/p}_{\varphi}[2]$$

is a trivial extension.

Compare $\underline{\widetilde{C_*}}(S^\infty)$ with the $\mathbb{Z}/p$-spectrum $S^\infty\alpha \wedge H\underline{\mathbb{Z}}$, which is also an $S^{\infty\alpha}$-module. The fixed points functor $(-)^{\mathbb{Z}/p}$ gives an equivalence $S^{\infty\alpha}$-Mod $\simeq$ Spectra, where

$$S^{\infty\alpha} \wedge H\underline{\mathbb{Z}} \longmapsto H\mathbb{Z}/p \vee \Sigma^2 H\mathbb{Z}/p \vee \Sigma^4 H\mathbb{Z}/p \vee \cdots .$$

So in the standard t-structure on $\mathbb{Z}/p$-equivariant spectra (with the heart being Mackey functors),

$$\tau_{\leq 2}(S^{\infty\alpha} \wedge H\underline{\mathbb{Z}}) = H\underline{\mathbb{Z}/p}_{\varphi} \vee H\underline{\mathbb{Z}/p}_{\varphi}[2]$$

is a trivial extension.

Conclusion: The forgetful functor

$$D(\underline{\mathbb{Z}}\text{-Mod}) \longrightarrow D(\mathbb{Z}/p\text{-Spectra})$$

is not faithful!

The quasi-smooth semiperfect case

What we really want: Mackey model for $THH_{\mathbb{Z}/p^r}(R)$, for R smooth over $\mathbb{F}_p$.

The quasi-smooth semiperfect case

What we really want: Mackey model for $THH_{\mathbb{Z}/p^r}(R)$, for R smooth over $\mathbb{F}_p$.

Theorem (Hesselholt)

For a smooth algebra R over $\mathbb{F}_p$,

$$THH(R)_*^{\mathbb{Z}/p^k} = \Omega W_{k+1}(R)[\sigma_k]$$

where $\Omega W_{k+1}(R)$ is Illusie's DeRham-Witt complex of length $k+1$.

For R a $\mathbb{F}_p$-algebra, the *(colimit) perfection* of R is

$$R_{perf} = \operatorname{colim}(R \to R \to R \to \cdots)$$

where all maps are the Frobenius φ.

For R a $\mathbb{F}_p$-algebra, the *(colimit) perfection* of R is

$$R_{perf} = \mathrm{colim}(R \to R \to R \to \cdots)$$

where all maps are the Frobenius φ.

Then $THH(R)$ has semiperfect descent:

$$THH(R) \to \left| THH(R_{perf} \otimes_R R_{perf} \otimes_R \cdots \otimes_R R_{perf}) \right|$$

is an equivalence. Each stage $R_{perf} \otimes_R \cdots \otimes_R R_{perf}$ of the cosimplicial resolution is *quasi-smooth* and semiperfect.

The derived cotangent complex

Quillen: for a map of commutative rings $A \to B$, the *derived cotangent complex*

$$L_{B/A} = B \otimes_P \Omega_{P/A}$$

(where P is a projective resolution of B over A) is the left derived functor of the Kähler differentials $\Omega_{B/A}$.

The derived cotangent complex

Quillen: for a map of commutative rings $A \to B$, the *derived cotangent complex*

$$L_{B/A} = B \otimes_P \Omega_{P/A}$$

(where P is a projective resolution of B over A) is the left derived functor of the Kähler differentials $\Omega_{B/A}$.

For $A \to B \to C$, there is a cofibration in the derived category of C-modules

$$C \otimes_B L_{B/A} \to L_{C/A} \to L_{C/B}.$$

The derived cotangent complex

Quillen: for a map of commutative rings $A \to B$, the *derived cotangent complex*

$$L_{B/A} = B \otimes_P \Omega_{P/A}$$

(where P is a projective resolution of B over A) is the left derived functor of the Kähler differentials $\Omega_{B/A}$.

For $A \to B \to C$, there is a cofibration in the derived category of C-modules

$$C \otimes_B L_{B/A} \to L_{C/A} \to L_{C/B}.$$

Theorem (Quillen)

For $A \to B$, $B \otimes_A B \simeq B$, there is a spectral sequence

$$E^2_{p,q} = H_{p+q}(Sym^q_B L_{B/A}) \Rightarrow Tor^A_{p+q}(B, B).$$

In particular, $H_1(L_{B/A}) = Tor^A_1(B, B)$.

The limit perfection S_R

Let R be a semiperfect $\mathbb{F}_p$ algebra. Its limit perfection is

$$S_R = \lim(\cdots \to R \to R \to R).$$

Let $J = Ker(S_R \to R)$. The maps $\mathbb{F}_p \to S_R \to R$ gives

$$L_{R/\mathbb{F}_p} \cong L_{R/S_R}.$$

Apply Quillen's theorem for $S_R \to R$ gives

$$H_1(L_{R/\mathbb{F}_p}) = Tor_1^{S_R}(R, R) = J/J^2.$$

The limit perfection S_R

Let R be a semiperfect $\mathbb{F}_p$ algebra. Its limit perfection is

$$S_R = \lim(\cdots \to R \to R \to R).$$

Let $J = Ker(S_R \to R)$. The maps $\mathbb{F}_p \to S_R \to R$ gives

$$L_{R/\mathbb{F}_p} \cong L_{R/S_R}.$$

Apply Quillen's theorem for $S_R \to R$ gives

$$H_1(L_{R/\mathbb{F}_p}) = Tor_1^{S_R}(R, R) = J/J^2.$$

A semiperfect $\mathbb{F}_p$-algebra R is *quasi-smooth* if

- $H_1(L_{R/\mathbb{F}_p}) = J/J^2$ is a free R-module of finite rank
- $H_n(L_{R/\mathbb{F}_p}) = 0$ for $n \neq 1$
- $Sym_R^q(J/J^2) = J^q/J^{q+1}$

The r=0 case

Let R be quasi-smooth semiperfect. The cofiber sequence of derived cotangent complexes for $\mathbb{F}_p \to R \otimes_{\mathbb{F}_p} R \to R$ gives

$$L_{R/R\otimes R} \simeq J/J^2[2].$$

The Quillen spectral sequence

$$H_*(Sym^q(L_{R/R\otimes R})) = Sym(J/J^2[2]) \Rightarrow Tor^{R\otimes R}(R, R)$$

collapses.

Next, analogously as in the case of $\mathbb{F}_p$, we have spectral sequence

$$Tor^{A_*\otimes R\otimes R}(R, R) = Tor^{A_*}(\mathbb{F}_p, \mathbb{F}_p) \otimes Sym(J/J^2[2]) \Rightarrow THH(R)_*.$$

- For $p = 2$, the spectral sequence collapses
- For $p > 2$, there are differentials similarly as in the $\mathbb{F}_p$-case by Bökstedt

- For $p = 2$, the spectral sequence collapses
- For $p > 2$, there are differentials similarly as in the $\mathbb{F}_p$-case by Bökstedt

Proposition (Burghardt, H., Kriz, Somberg)

For R quasi-smooth semiperfect,

$$THH(R)_{2q} = \bigoplus_{i=0}^{q} J^i/J^{i+1}$$

$$THH(R)_{2q+1} = 0.$$

The general r case

- Choose basis B for J/J^2 over R
- Let B_q be the set of unordered q-tuples of B-elements, i. e. B_q is a basis for $J^q/J^{q+1} = Sym^q(J/J^2)$
- Write
$$W_r^{B_q}(R) = \bigoplus_{B_q} W_r(R)$$
- Also, choose a set of integral lifts $\widetilde{B}$ of B via $W(S_R) \to R$. So we also get integral lifts $\widetilde{B_q}$.
- The $\bigoplus_{\widetilde{B_q}} \mathcal{W}_{r,j}$ are functorial in the category of $\underline{\mathbb{Z}_p}$-modules (on the nose)

Write

$$\mathcal{W}_r^{\widetilde{B_q}} = \lim \left(\begin{array}{ccccc} & & & & \bigoplus_{\widetilde{B_q}} \mathcal{W}_{r,r} \\ & & & & \big\downarrow{\scriptstyle \varphi} \\ & & \cdots & \longrightarrow & \bigoplus_{\widetilde{B_q}} \mathcal{W}_{r,r} \\ & & \big\downarrow & & \\ & & \bigoplus_{\widetilde{B_q}} \mathcal{W}_{r,1} \xrightarrow[\pi]{} & \cdots & \\ & & \big\downarrow{\scriptstyle \varphi} & & \\ \bigoplus_{\widetilde{B_q}} \mathcal{W}_{r,0} & \xrightarrow[\pi]{} & \bigoplus_{\widetilde{B_q}} \mathcal{W}_{r,1} & & \end{array} \right)$$

The vertical maps are induced by the Frobenius φ on R. The horizontal maps are projections.

The case of general r

Theorem (Burghardt, H., Kriz, Somberg)

For a quasi-smooth semiperfect $\mathbb{F}_p$-algebra R,

$$THH^{\mathbb{Z}/p^r}(R)_{2q} = \bigoplus_{i=0}^{q} W^{B_i}_{r+1}(R)$$

$$THH^{\mathbb{Z}/p^r}(R)_{2q+1} = 0.$$

In the derived category of $\underline{\mathbb{Z}_p}$-modules, the $\mathbb{Z}/p^r$-equivariant

$$THH_{\mathbb{Z}/p^r}(R) = \bigoplus_{q \geq 0} \mathcal{W}_R^{\widetilde{B_q}}[2q].$$

Again, induction on r using the Tate diagram and the cyclotomic condition.