May Spectral Sequence and Minimal Resolutions

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IWOAT 2025 @SUSTech
March 3, 2025

May spectral sequence

Let $\mathcal{A}$ be the Steenrod algebra at the prime 2.
Adams spectral sequence:

$$H^{s,t}(\mathcal{A}) = \mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \implies \pi_{t-s}(\mathbb{S}_2^\wedge)$$

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May spectral sequence:

$$H^{s,t,u}(E^0\mathcal{A}) = \mathrm{Ext}_{E^0\mathcal{A}}^{s,t,u}(\mathbb{F}_2, \mathbb{F}_2) \implies \mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$$

May spectral sequence

The E_1 page of the May spectral sequence is given by

$$E_1 = \mathbb{F}_2[R^i_j : i \geq 0, j > 0]$$

where R^i_j corresponds to $\xi_j^{2^i}$ in the dual Steenrod algebra and

$$d_1 R^i_j = \sum_{k=1}^{j-1} R^{i+k}_{j-k} R^i_k$$

$$|R^i_j|_{(s,t,w)} = (1, 2^i(2^j - 1), j)$$

May spectral sequence

If we use the generators to form a matrix:

$$R = \begin{pmatrix} 0 & R_1^0 & R_2^0 & R_3^0 & R_4^0 & R_5^0 & R_6^0 & \cdots \\ 0 & 0 & R_1^1 & R_2^1 & R_3^1 & R_4^1 & R_5^1 & \cdots \\ 0 & 0 & 0 & R_1^2 & R_2^2 & R_3^2 & R_4^2 & \cdots \\ 0 & 0 & 0 & 0 & R_1^3 & R_2^3 & R_3^3 & \cdots \\ 0 & 0 & 0 & 0 & 0 & R_1^4 & R_2^4 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & R_1^5 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

then the differentials can be written as $dR = R^2$.

May spectral sequence

Conjecture (May)

The elements $h_i(S)$, b_{ij} form an additive basis of the indecomposables of E_2.

May has proved that these are indecomposables.

However, we still do not know if they generate the whole E_2-page.

May spectral sequence

Conjecture (L.)

The May E_2-page is generated by $h_i(S)$, b_{ij} with the following relations

1. $\sum_k b_{ik} b_{kj} = 0$
2. $h_{S_1,T_1} h_{S_2,T_2} = 0$ with some conditions on S_i, T_i.
3. $\sum_{s\in S} b_{sj} h_{S-\{s\},T+\{s\}} = 0, \dots$
4. $\sum_{t\in T} b_{it} h_{S+\{t\},T-\{t\}} = 0, \dots$
5. $h_{S_1,T_1} h_{S_2,T_2} = \sum_{I\subset T_1''\cap S_2} h_{S_1''+I,T_1''-I} h_{S_1'+S_2-I,T_1'+T_2+I}, \cdots$
6. $h_{S_1,T_1} h_{S_2,T_2} = \sum_{I\subset T_1\cap S_2''} h_{S_2''-I,T_2''+I} h_{S_1+S_2'+I,T_1+T_2'-I}, \cdots$
7. $h_{S_1,T_1} h_{S_2,T_2} = \sum_{\substack{I\subset S_1' \\ J\subset T_2'}} h_{S_1'-I,T_1'+I} h_{S_2'+J,T_1'-J} b_{S_1''+I,T_2''+J}, \cdots$
8. $h'_{S_1,T_1} \prod x_i = h'_{S_2,T_2} \prod y_i$ with some conditions implies $h_{S_1,T_1} \prod x_i = h_{S_2,T_2} \prod y_i$.

May spectral sequence

Conjecture (L.)

${}^{May}E_2$ is nilpotent free.

Theorem (L.)

For all $i \geq 0$ and $x \in {}^{May} E_2$, $h_i x \neq 0$ implies $h_i^k x \neq 0$ for all $k \geq 0$.

May spectral sequence

Theorem (May)

$d_2(b_{ij}) = h_{i+1} b_{i+1,j} + h_{j+1} b_{i,j-1}$.

Example: $d_2 b_{02} = h_1^3 + h_0^2 h_2$.

Theorem (May-L.)

The d_2 differentials of $h_i(S)$ are given by

$$d_2 h_i(s_1, \ldots, s_{n-1}) = \sum_{\substack{j=n-1 \text{ or} \\ s_j+1<s_{j+1}}} h_{i+s_j+1} h_i(s_1, \ldots, s_{j-1}, s_j+1, s_{j+1}, \ldots, s_{n-1}).$$

Example: $d_2 h_i(0) = h_0 h_2^2$

May spectral sequence

It has been checked by computer programs that all the conjectures mentioned hold in stems ≤ 1000.

Theorem (L.)

If X is a **commutative differential graded algebra** over $\mathbb{F}_p$ that is large as a $\mathbb{F}_p$-vector space but small in terms of generators and relations, then there is an efficient algorithm based on **Gröbner basis** computing HX with generators and relations as the output.

I applied this algorithm to the E_1-page of the May spectral sequence and obtained the generators and relations of the E_2-page in a big range.

May spectral sequence

Questions

How to compute higher May differentials stem by stem using a computer?

Minimal resolutions

In order to compute the E_2-page of the Adams spectral sequence

$$\mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2),$$

we can also use a minimal resolution of $\mathbb{F}_2$

$$\mathbb{F}_2 \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots$$

where P_i are free $\mathcal{A}$-modules.

Minimal resolutions

To obtain a minimal resolution, we need to compute kernels of linear maps of the following form

$$\mathcal{A}^n \to \mathcal{A}^m.$$

We need to solve systems of linear equations with coefficients in $\mathcal{A}$.

If we replace $\mathcal{A}$ by a field, this is the stand linear algebra.

If we replace $\mathcal{A}$ by a commutative ring, we can use the classical Gröbner basis to solve these systems of equations.

However, the Steenrod algebra $\mathcal{A}$ is noncommutative.

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Theorem (L.)

There is a noncommutative generalization of Gröbner basis and associated algorithms which apply to the Steenrod algebra.

Minimal resolutions

The algorithms apply to finitely-filtered-commutative algebras including primitively generated Hopf algebras such as the Steenrod algebra.

We can use the new algorithms to implement efficient computer programs and obtain the following.

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Theorem (L.)

The bigraded algebra

$$\bigoplus_{t\leq 261} \mathrm{Ext}_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$$

in internal degrees up to 261 is an algebra with 2914 indecomposables, 23822 basis elements and 227498 indecomposable relations. The complete lists are given in
`https://doi.org/10.5281/zenodo.7786289`.

May spectral sequence and Minimal resolutions

Questions 1

How can we compare the Ext groups computed using a minimal resolution to elements in the May spectral sequence?

Example: $c_0 \in$ AdamsE2 is represented by $h_0(1)h_1$ in MaySS.

May spectral sequence and Minimal resolutions

Questions 1

How can we compare the Ext groups computed using a minimal resolution to elements in the May spectral sequence?

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Questions 2

Why do we want to compare them?

The May spectral sequence gives us more structural understanding of the Ext groups and it provides a lot of information about the whole E_2-page.

Application: $\mathrm{Ext}(X) \to v_n^{-1}\mathrm{Ext}(X)$.

May spectral sequence and Minimal resolutions

Compute a free resolution of an $\mathcal{A}$-module.

$$\cdots \longleftarrow P_s \xleftarrow{d_{s+1}} P_{s+1} \longleftarrow P_{s+2} \longleftarrow \cdots$$

P_s	P_{s+1}	P_{s+2}
$d_{s+1} a_{s+1,1}$	$a_{s+1,1}$	
$\vdots$	$\vdots$	
$d_{s+1} a_{s+1,k}$	$a_{s+1,k}$	
linear combinations	linear combinations	
	linear combinations	$a_{s+2,1}$
	$\vdots$	$\vdots$

May spectral sequence and Minimal resolutions

The E_2-page of MaySS can be also computed by a minimal resolution over $E^0\mathcal{A}$.

$$\cdots \longleftarrow (E^0\mathcal{A})^{n_s} \xleftarrow{d_{s+1}} (E^0\mathcal{A})^{n_{s+1}} \longleftarrow (E^0\mathcal{A})^{n_{s+2}} \longleftarrow \cdots$$

P_s	P_{s+1}	P_{s+2}
$d_{s+1}b_{s+1,1}$	$b_{s+1,1}$	
$\vdots$	$\vdots$	
$d_{s+1}b_{s+1,l}$	$b_{s+1,l}$	
linear combinations	linear combinations	
	linear combinations	$b_{s+2,1}$
	$\vdots$	$\vdots$

May spectral sequence and Minimal resolutions

We can actually lift the minimal resolution over $E^0\mathcal{A}$ to a resolution over $\mathcal{A}$ which is *not minimal*.

$$\cdots \longleftarrow (E^0\mathcal{A})^{n_s} \xleftarrow{d_{s+1}} (E^0\mathcal{A})^{n_{s+1}} \longleftarrow (E^0\mathcal{A})^{n_{s+2}} \longleftarrow \cdots$$

$$\cdots \longleftarrow \mathcal{A}^{n_s} \xleftarrow{d_{s+1}} \mathcal{A}^{n_{s+1}} \longleftarrow \mathcal{A}^{n_{s+2}} \longleftarrow \cdots$$

P_s	P_{s+1}	P_{s+2}
$d_{s+1}b_{s+1,1}$ + higher filtrations	$b_{s+1,1}$	
$\vdots$	$\vdots$	
$d_{s+1}b_{s+1,l}$ + higher filtrations	$b_{s+1,l}$	
linear combinations	linear combinations	
	linear combinations	$b_{s+2,1}$
	$\vdots$	$\vdots$

Question

How to fill these higher filtrations effectively?

Picking a lift of an $E^0\mathcal{A}$-resolution is not canonical. In fact from a lift we can obtain a May E_2-page with noncanonical "total differentials" which encode all May differentials.

May spectral sequence and Minimal resolutions

Question

How to fill these higher filtrations effectively?

Picking a lift of an $E^0\mathcal{A}$-resolution is not canonical. In fact from a lift we can obtain a May E_2-page with noncanonical "total differentials" which encode all May differentials.

Solution

We start with a minimal resolution over $\mathcal{A}$. Whenever we have an entry in the P_s-column starting with terms in the table for $E^0\mathcal{A}$, we add additional rows to the table and make the table exact again.

May spectral sequence and Minimal resolutions

Compute a free resolution of an $\mathcal{A}$-module.

$$\cdots \longleftarrow P_s \xleftarrow{d_{s+1}} P_{s+1} \longleftarrow P_{s+2} \longleftarrow \cdots$$

P_s	P_{s+1}	P_{s+2}
$d_{s+1}a_{s+1,1}$	$a_{s+1,1}$	
$\vdots$	$\vdots$	
$d_{s+1}a_{s+1,k}$	$a_{s+1,k}$	
$d_{s+1}b_{s+1,1}$ + higher filtrations	a'	
$d_{s+1}b_{s+1,1}$ + higher filtrations	$b_{s+1,1}$	
$\vdots$	$\vdots$	
	$b_{s+1,1} + a'$	e
	$\vdots$	$\vdots$

May spectral sequence and Minimal resolutions

In the end, we obtain a resolution which contains

1. the minimal resolution over $\mathcal{A}$ as a subcomplex,
2. a lift of the minimal resolution over $E^0\mathcal{A}$ as a subcomplex, and
3. the comparing map.

These data tell us all the May differentials in the computed range and the representing elements in the May spectral sequence for elements in the E_2-page of the Adams spectral sequence.

For the sphere, this method might not go very far (up to dimension 150?). But for some spectra with sparse May E_2-page we can go much further.

Thank You!