Periodicity, vanishing lines, and transchromatic phenomena in chromatic homotopy theory

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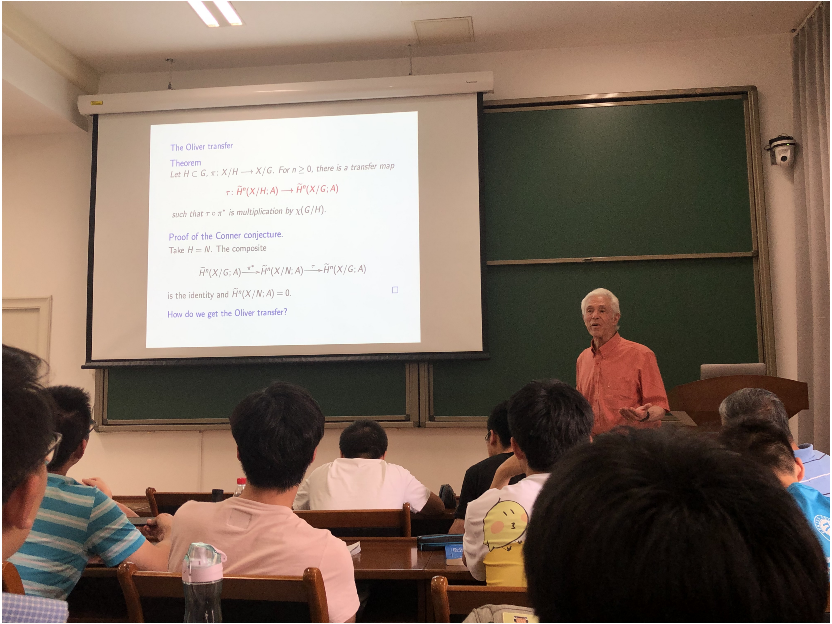
The Oliver transfer
Theorem
Let $H \subset G$, $\pi: X/H \longrightarrow X/G$. For $n \geq 0$, there is a transfer map
$\tau: \widetilde{H}^n(X/H; A) \longrightarrow \widetilde{H}^n(X/G; A)$
such that $\tau \circ \pi^*$ is multiplication by $\chi(G/H)$.
Proof of the Conner conjecture.
Take $H = N$. The composite
$\widetilde{H}^n(X/G; A) \xrightarrow{\pi^*} \widetilde{H}^n(X/N; A) \xrightarrow{\tau} \widetilde{H}^n(X/G; A)$
is the identity and $\widetilde{H}^n(X/N; A) = 0$.
How do we get the Oliver transfer?


E=MC2
BLACK PINK
KILL THIS LOVE


太和充滿
ARMANI EXCHANGE
FUSC

The Kervaire invariant problem

Theorem (Hill–Hopkins–Ravenel 2009)

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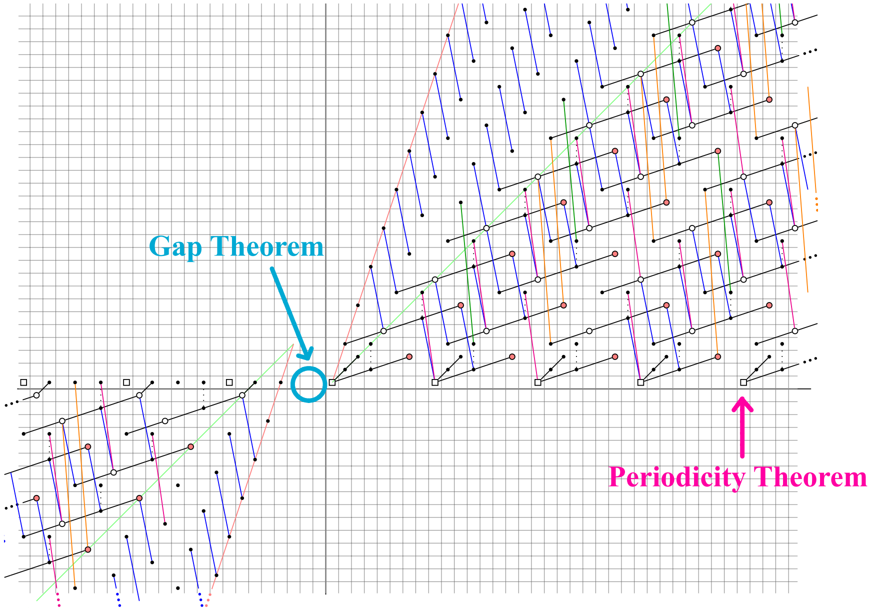
- Gap Theorem: $\pi_i(D^{-1}MU^{((C_8))})^{C_8} = 0$ for $i = -1, -2, -3$.

Motivational Picture



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- Question: what meaningful phenomena can we extract from the slice spectral sequence?
- Question: what do they tell us about equivariant and chromatic homotopy theory?

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- $\mathcal{S}_{\leq n} = \{X \mid Map_G(Y, X) \simeq *, Y \in \mathcal{S}_{>n}\}$

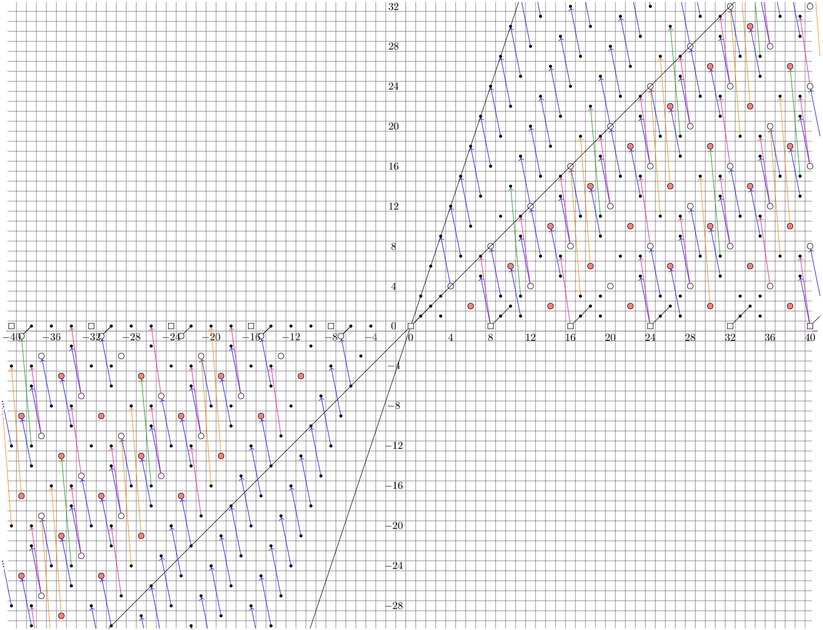
The slice spectral sequence

Slice tower:

$$\begin{array}{ccccccccccc} X & \longrightarrow & \cdots & \longrightarrow & P^{n+1}X & \longrightarrow & P^nX & \longrightarrow & P^{n-1}X & \longrightarrow & \cdots \\ & & & & \uparrow & & \uparrow & & \uparrow & & \\ & & & & P^{n+1}_{n+1}X & & P^n_nX & & P^{n-1}_{n-1}X & & \end{array}$$

$$E_2^{s,t} = \underline{\pi}_{t-s} P_t^t X \Longrightarrow \underline{\pi}_{t-s} X$$

Example: a C_4-slice spectral sequence



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- Question: studying the entire slice spectral sequence all at once is hard, but... can we isolate each region and study it separately?
- Answer: the generalized Tate diagram

Tate diagram

Greenlees and May introduced the Tate diagram for any G-spectrum X:

$$\begin{array}{ccccc} EG_+ \wedge X & \longrightarrow & X & \longrightarrow & \widetilde{E}G \wedge X \\ \downarrow{\simeq} & & \downarrow & & \downarrow \\ EG_+ \wedge F(EG_+, X) & \longrightarrow & F(EG_+, X) & \longrightarrow & \widetilde{E}G \wedge F(EG_+, X) \end{array}$$

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- The right square is a pullback square (Tate square)
- This is a powerful tool in equivariant homotopy theory

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$$\begin{array}{ccccc} X_{hG} & \longrightarrow & X^G & \longrightarrow & (\widetilde{E}G \wedge X)^G \\ \downarrow \simeq & & \downarrow & & \downarrow \\ X_{hG} & \longrightarrow & X^{hG} & \longrightarrow & X^{tG} \end{array}$$

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- We can decompose X into "easier pieces", analyze them individually, and "glue" them back using the Tate diagram

Generalized Tate diagram

- For $\mathcal{F}$ a family of subgroups of G, Greenlees and May further introduced the generalized Tate diagram

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- $E\mathcal{F}$ is the unique G-space characterized by the property

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- When $\mathcal{F} = \{e\}$, $E\mathcal{F} = EG$, and we get the classical Tate diagram

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- Example: $P_\bullet X \longrightarrow \widetilde{E}\mathcal{F} \wedge P_\bullet X$

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- For N a normal subgroup of G, set $\mathcal{F} = \mathcal{F}[N]$: family of all subgroups of G not containing N
- The localized slice spectral sequence converges strongly to the homotopy groups of $(\widetilde{E}\mathcal{F}[N] \wedge X)^G = \Phi^N(X)^{G/N}$

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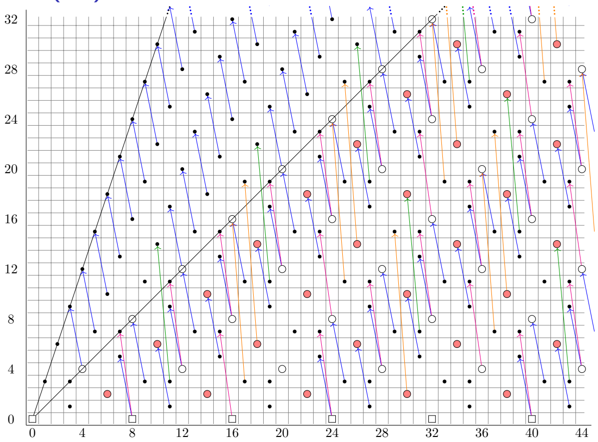
$N = C_4$: $\Phi^{C_4}(X)$



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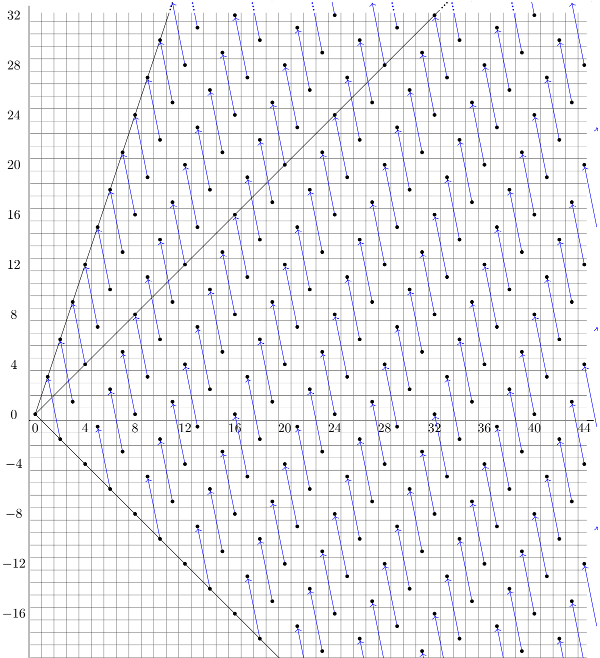

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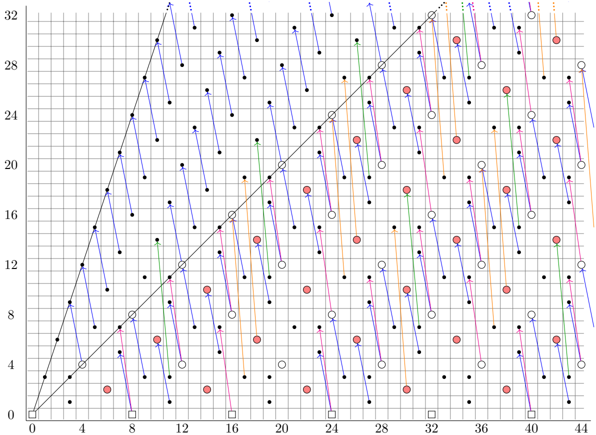

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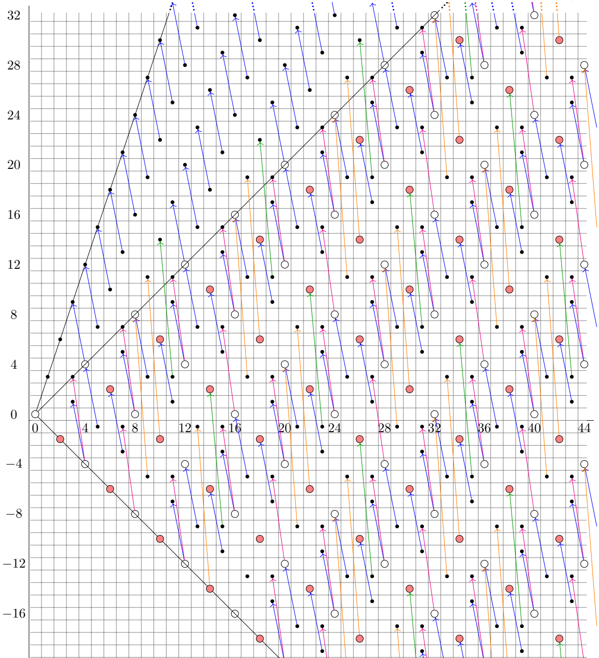

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- Observation 3:
  The same holds for the negative cone, by using the spectral sequence $F(\widetilde{E}\mathcal{F}, P_\bullet X)$ from the generalized Tate diagram

Slice Recovery Theorem

Theorem (Meier–S.–Zeng, Liu–S.–Yan)

1. *In the positive cone, the map*

$$\mathrm{SliceSS}(X) \longrightarrow \widetilde{E}\mathcal{F}[N] \wedge \mathrm{SliceSS}(X)$$

induces an isomorphism of spectral sequences on or above the line of slope $(h-1)$*. Here,* h *is the order of the largest subgroup in* $\mathcal{F}[N]$*.*

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2. *In the negative cone, the map*

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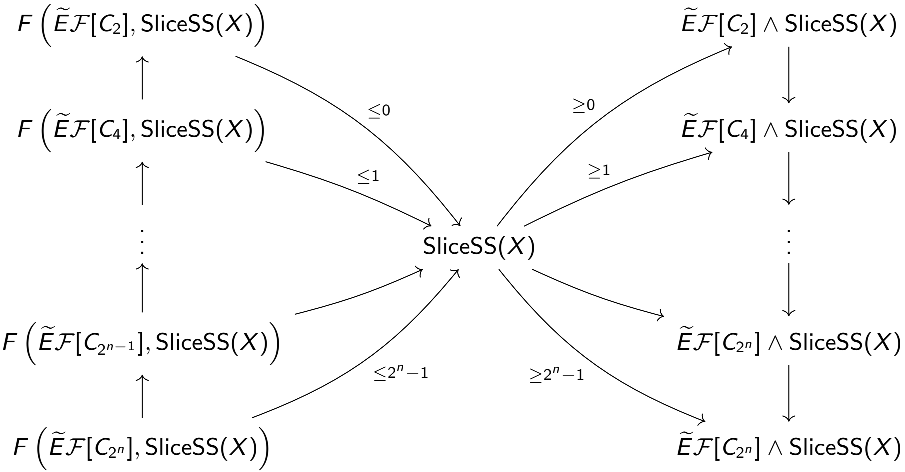

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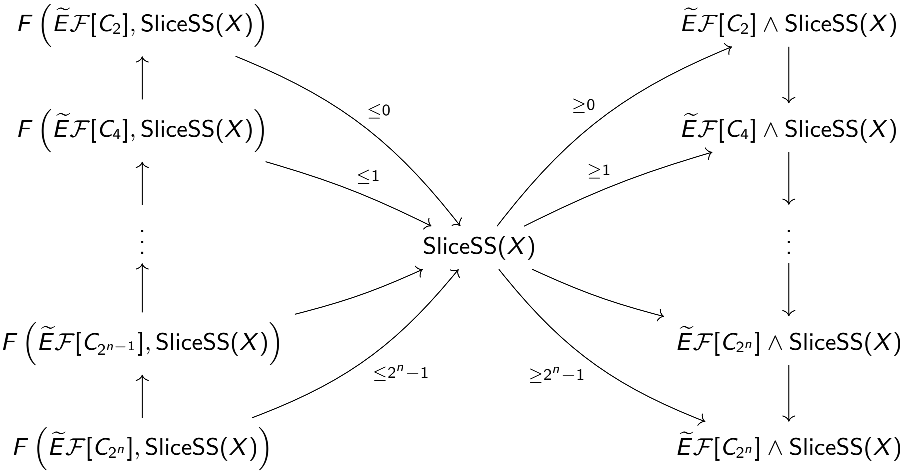

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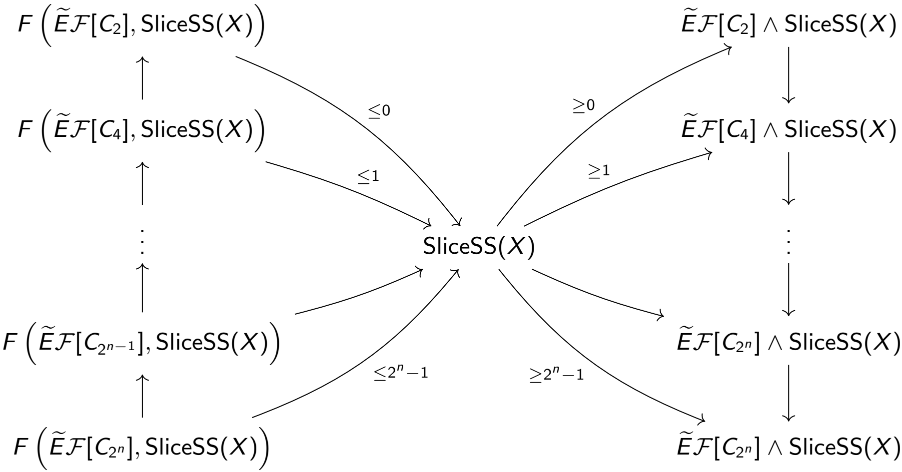

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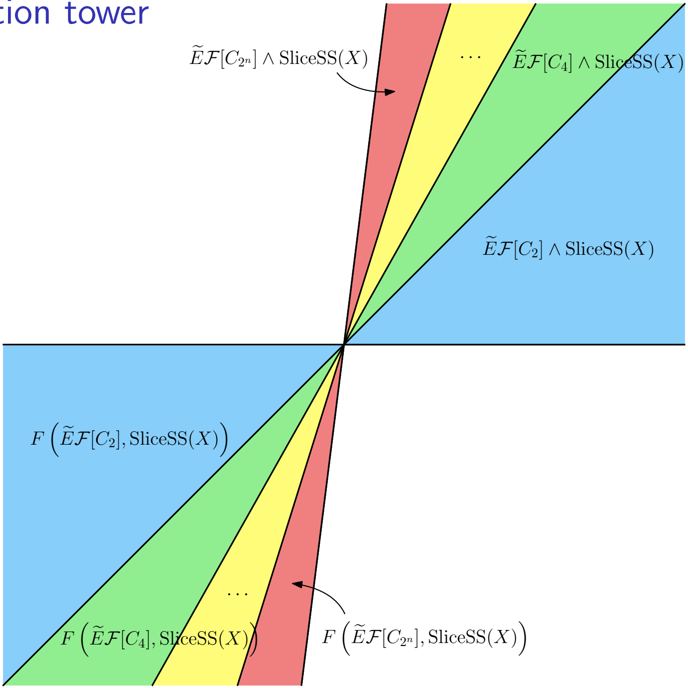
- Remark: we can always form such a tower for any equivariant spectral sequence, but not guaranteed to have a recovery theorem

Stratification tower



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- 2-locally, $BP^{((C_{2^n}))} := N_{C_2}^{C_{2^n}}(BP_{\mathbb{R}})$

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- 2-locally, $BP^{((C_{2^n}))} := N_{C_2}^{C_{2^n}}(BP_{\mathbb{R}})$
- These theories are very useful for resolving the Kervaire invariant problem and for studying Lubin–Tate theories

Lubin–Tate theories

Non-equivariantly: the formal group laws associated with $BP\langle n\rangle$ give models for E_n

$$\begin{array}{ccccccc}
BP \longrightarrow \cdots \longrightarrow & BP\langle 3\rangle & \longrightarrow & BP\langle 2\rangle & \longrightarrow & BP\langle 1\rangle \\
& \downarrow & & \downarrow & & \downarrow \\
& v_3^{-1}BP\langle 3\rangle & & v_2^{-1}BP\langle 2\rangle & & v_1^{-1}BP\langle 1\rangle \\
& \rightsquigarrow & & \rightsquigarrow & & \rightsquigarrow \\
& E_3 & & E_2 & & E_1
\end{array}$$

Quotients of $BP^{(\!(C_{2^n})\!)}$

- C_{2^n}-equivariantly, we have $BP^{(\!(C_{2^n})\!)}$ instead of BP, and we can form equivariant quotients $BP^{(\!(C_{2^n})\!)}\langle m\rangle$ (Hill–Hopkins–Ravenel)

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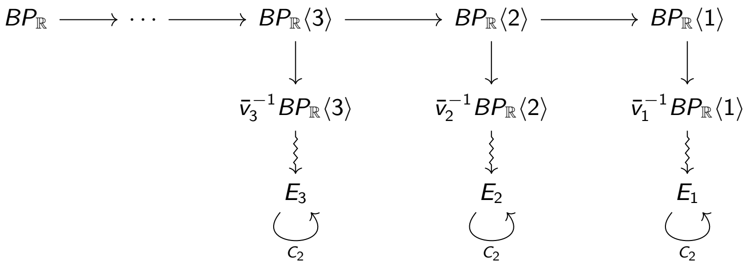
- These quotients have good slices
  - Odd slices $\simeq *$
  - Computable even slices $\Longrightarrow$ computable E_2-page

Models of Lubin–Tate theory

$G = C_2$: the C_2-equivariant formal group laws associated with $BP_{\mathbb{R}}\langle n\rangle$ give C_2-equivariant models for E_n



Models of Lubin–Tate theory

$G = C_4$: the C_4-equivariant formal group laws associated with $BP^{(\!(C_4)\!)}\langle n\rangle$ give C_4-equivariant models for E_{2n}

$$\begin{array}{ccccccc}
BP^{(\!(C_4)\!)} & \longrightarrow & BP^{(\!(C_4)\!)}\langle 3\rangle & \longrightarrow & BP^{(\!(C_4)\!)}\langle 2\rangle & \longrightarrow & BP^{(\!(C_4)\!)}\langle 1\rangle \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & D_6^{-1}BP^{(\!(C_4)\!)}\langle 3\rangle & & D_4^{-1}BP^{(\!(C_4)\!)}\langle 2\rangle & & D_2^{-1}BP^{(\!(C_4)\!)}\langle 1\rangle \\
 & & \rightsquigarrow & & \rightsquigarrow & & \rightsquigarrow \\
 & & E_6 & & E_4 & & E_2 \\
 & & \circlearrowright C_4 & & \circlearrowright C_4 & & \circlearrowright C_4
\end{array}$$

Models of Lubin–Tate theory

Theorem (Beaudry–Hill–S.–Zeng)

The C_{2^n}-equivariant formal group laws associated with $BP^{(\!(C_{2^n})\!)}\langle m\rangle$ are of heights $(2^{n-1}\cdot m)$, and they give C_{2^n}-equivariant models of $E_{2^{n-1}\cdot m}$

$$
\begin{array}{ccccccc}
BP^{(\!(C_{2^n})\!)} & \longrightarrow & BP^{(\!(C_{2^n})\!)}\langle 3\rangle & \longrightarrow & BP^{(\!(C_{2^n})\!)}\langle 2\rangle & \longrightarrow & BP^{(\!(C_{2^n})\!)}\langle 1\rangle \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & D^{-1}_{3\cdot 2^{n-1}} BP^{(\!(C_{2^n})\!)}\langle 3\rangle & & D^{-1}_{2\cdot 2^{n-1}} BP^{(\!(C_{2^n})\!)}\langle 2\rangle & & D^{-1}_{2^{n-1}} BP^{(\!(C_{2^n})\!)}\langle 1\rangle \\
 & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\
 & & E_{3\cdot 2^{n-1}} & & E_{2\cdot 2^{n-1}} & & E_{2^{n-1}} \\
 & & \circlearrowleft C_{2^n} & & \circlearrowleft C_{2^n} & & \circlearrowleft C_{2^n}
\end{array}
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Models of Lubin–Tate theory

- For our models, each E_h is equipped with the C_{2^n}-orientation (Hahn–S.)

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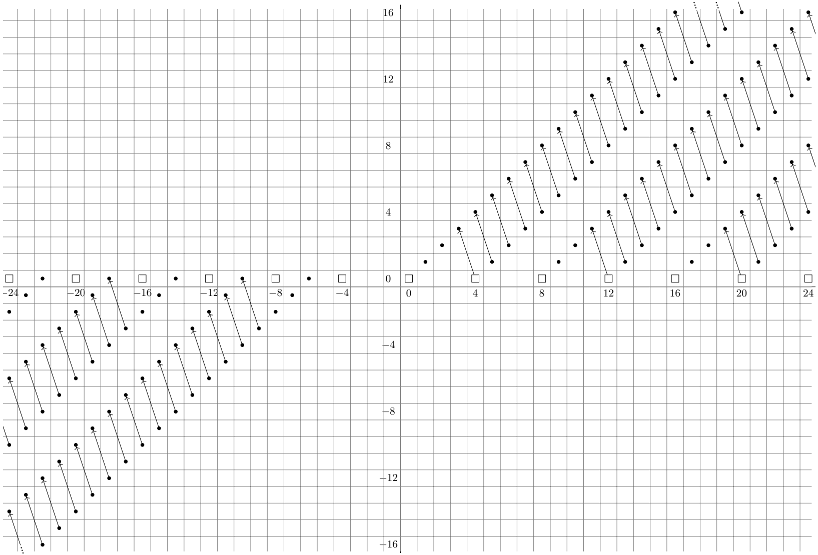
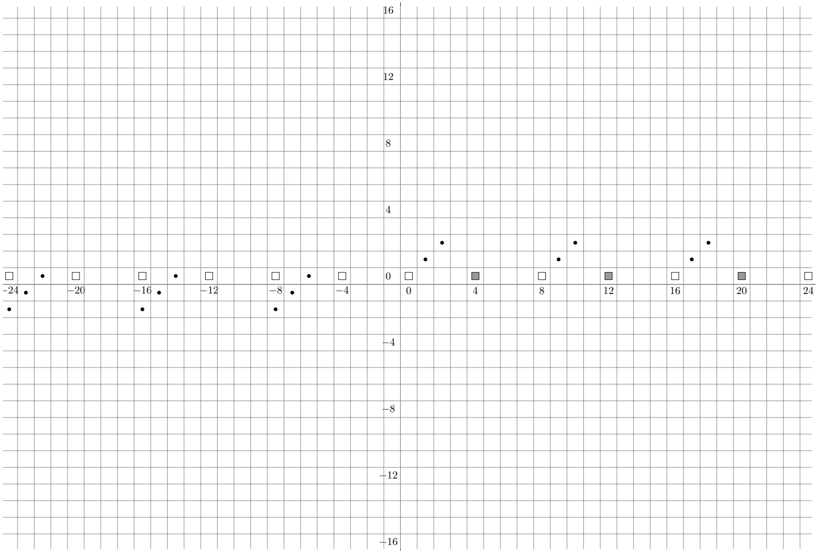
$$BP^{(\!(C_{2^n})\!)} \longrightarrow E_h$$
- This induces a map of spectral sequences, which is a quotient map on the E_2-page
- Upshot: we get equivariant "geometric models" for E_h, and they are great for doing computations

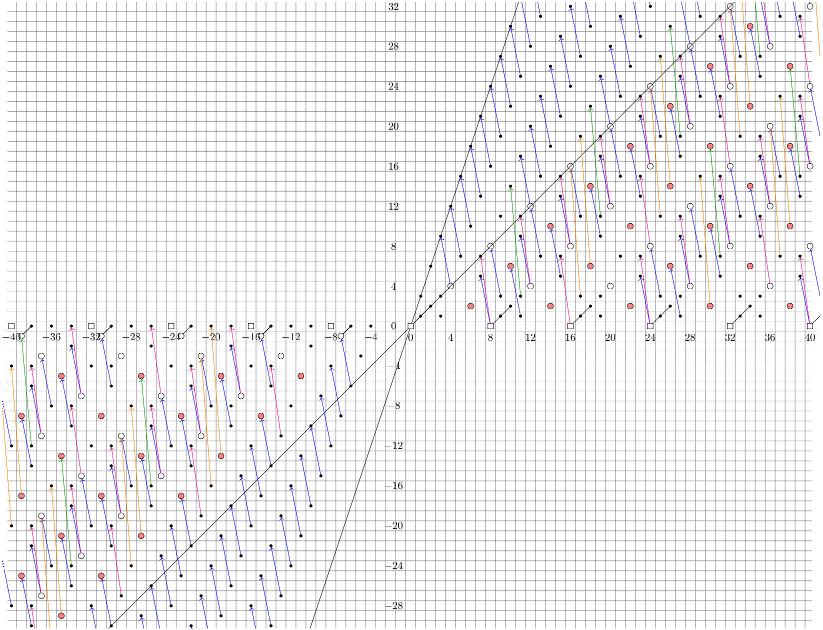
SliceSS($\bar{v}_1^{-1} BP_{\mathbb{R}}\langle 1\rangle$): Dugger ($E_1^{hC_2} = KO_2^{\wedge}$)



SliceSS($\bar{v}_1^{-1} BP_{\mathbb{R}}\langle 1\rangle$): E_∞-page



SliceSS($D_2^{-1}BP^{((C_4))}\langle 1\rangle$): Hill–Hopkins–Ravenel ($E_2^{hC_4}$)



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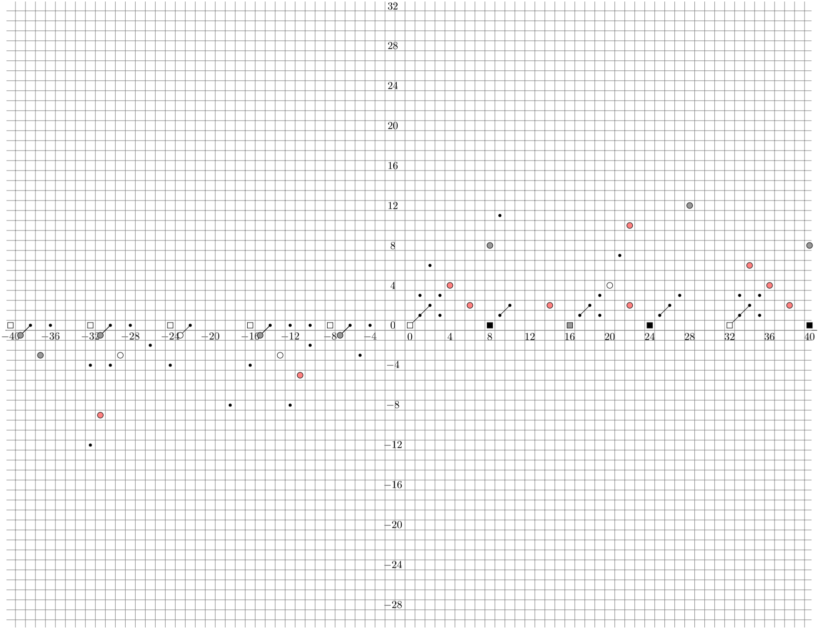
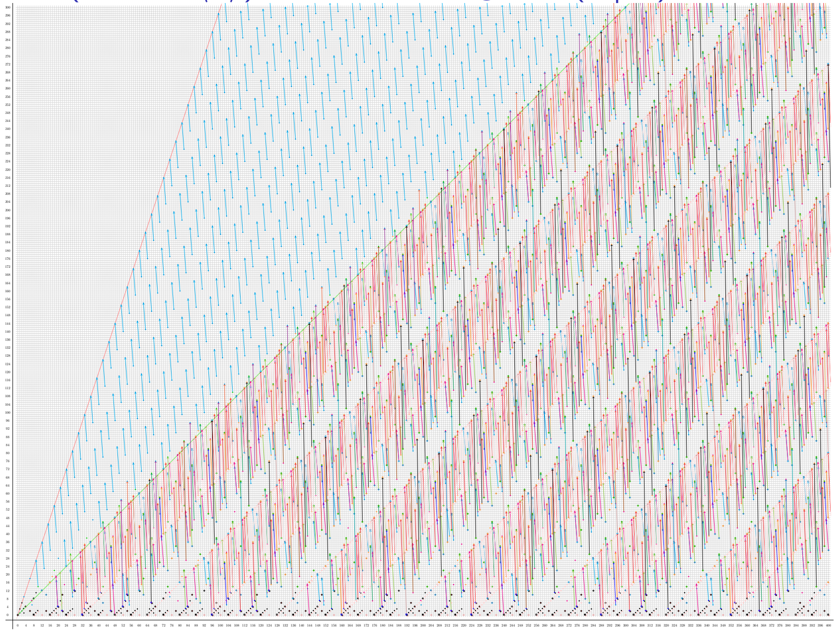
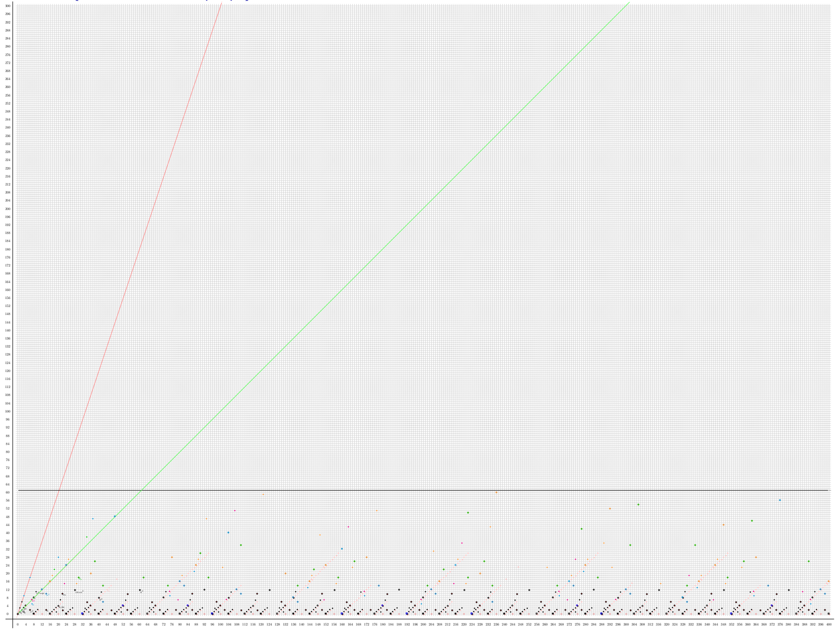


SliceSS($BP^{((C_4))}\langle 2\rangle$): Hill–S.–Wang–Xu ($E_4^{hC_4}$)



SliceSS($BP^{((C_4))}\langle 2\rangle$): E_∞-page



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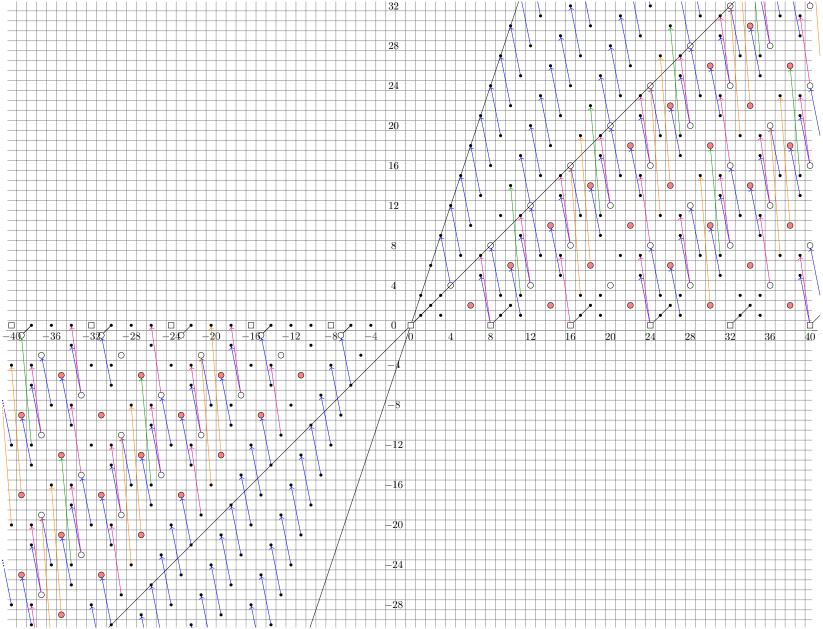
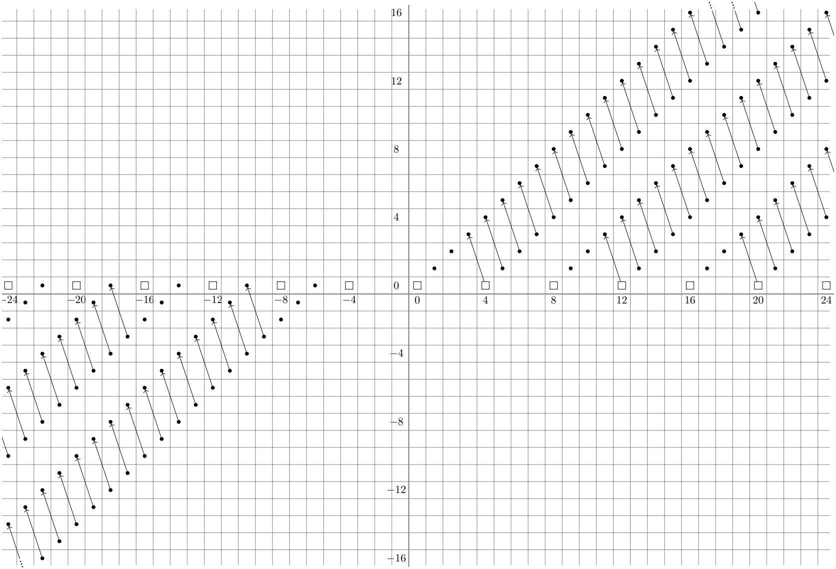
- Instead of computing them one at a time...
  Can we find generals patterns in SliceSS($BP^{(\!(G)\!)}\langle m \rangle$) across different groups and heights?
- As we vary G and m, how are the different $BP^{(\!(G)\!)}\langle m \rangle$ related to each other? Induction?

SliceSS($D_2^{-1} BP^{((C_4))}\langle 1 \rangle$): $E_2^{hC_4}$



SliceSS($\bar{v}_1^{-1} BP_{\mathbb{R}}\langle 1\rangle$): $E_1^{hC_2}$



Same differential pattern! $d_3 \leftrightsquigarrow d_5$

The Transchromatic Isomorphism Theorem

Theorem (Meier–S.–Zeng, Liu–S.–Yan)

There is a shearing isomorphism

$$\mathrm{SliceSS}\left(D^{-1}_{2^{n-1}\cdot m} BP^{(\!(C_{2^n})\!)}\langle m\rangle\right) \leftrightsquigarrow \mathrm{SliceSS}\left(D^{-1}_{2^{n-2}\cdot m} BP^{(\!(C_{2^{n-1}})\!)}\langle m\rangle\right)$$

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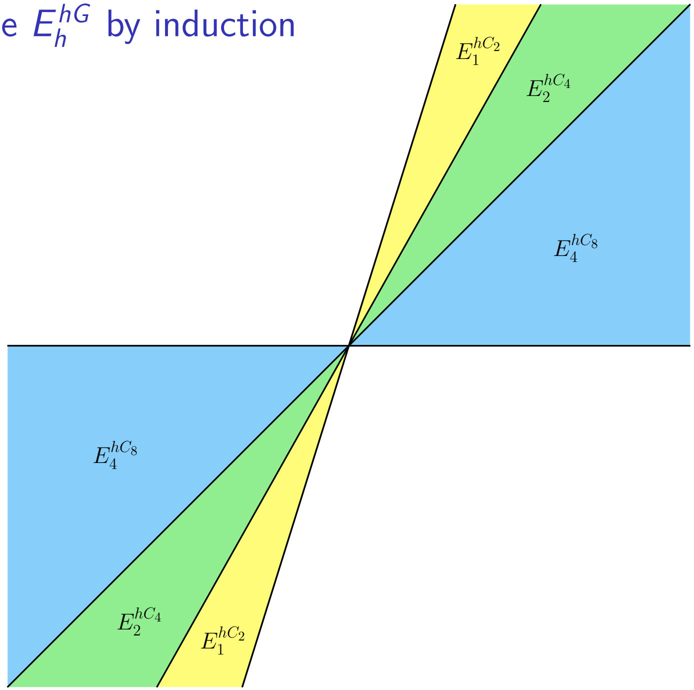
$$\mathrm{SliceSS}\left(E_h^{hG}\right) \leftrightsquigarrow \mathrm{SliceSS}\left(E_{h/2}^{h(G/C_2)}\right)$$

where $d_{2r-1} \leftrightsquigarrow d_r$.

Compute E_h^{hG} by induction



Vanishing lines

Theorem (Duan–Li–S.)

There is a horizontal vanishing line in $\mathrm{SliceSS}(E_h^{hC_{2^n}})$ *and* $\mathrm{HFPSS}(E_h^{hC_{2^n}})$ *of filtration* $2^{h+n} - 2^n + 1$*, and all the differentials are of lengths* $\leq 2^{h+n} - 2^n + 1$*.*

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- It is a consequence of the nilpotence theorem (Devinatz–Hopkins–Smith) that there are such horizontal vanishing lines at *some* finite filtration
- Theoretically useful, but in practice can't use the existence result to prove any differentials
- Having this precise vanishing line is very useful for computations

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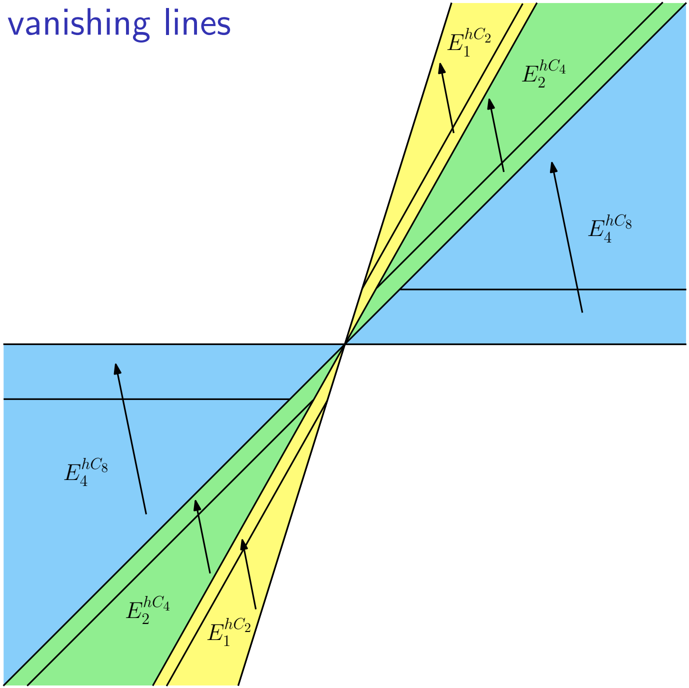
- Computationally, this imposes even more constraints on the possible differentials

Sheared vanishing lines



Periodicities for E_h^{hG}

- The theories E_h^{hG} are periodic
  - $E_1^{hC_2}$ is 8-periodic (real Bott periodicity)
  - $E_2^{hG_{24}}$ is 192-periodic ($K(2)$-local TMF)
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- Question: for any height h and finite group G, what is the periodicity of E_h^{hG}?

Periodicity Theorem

Theorem (Duan–Hill–Li–Liu–S.–Wang–Xu)

For any $h \geq 1$ and G a finite subgroup of $\mathbb{S}_h$, E_h^{hG} is $P_{h,G}$-periodic. Here, $P_{h,G} := \frac{|G|}{|H|} \cdot P_{h,H}$, where H is a 2-Sylow subgroup of G and

$$P_{h,H} := \begin{cases} 2 & \text{if } H = e \\ 2^{h+n+1} & \text{if } H = C_{2^n} \\ 2^{h+4} & \text{if } H = Q_8 \end{cases}$$

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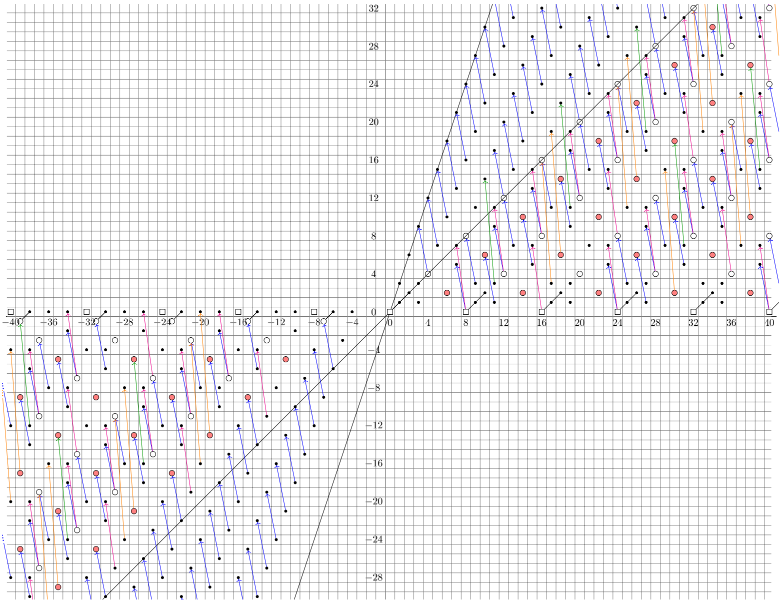
- This has very nice computational consequences
- If we know the end result is periodic beforehand, then we can force differentials



This entire computation can be determined by transchromatic isomorphism, vanishing lines, and periodicities

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  $\Longrightarrow E_2^{hC_4}$ is 32-periodic

$RO(G)$-periodicity lattice

Definition

$\mathbb{L}_{h,G} =$ *the free abelian subgroup of $RO(G)$ (under addition) that is generated by all V such that E_h is V-periodic*

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- Example: $\mathbb{L}_{2,C_4} = \mathbb{Z}\langle \rho_4 \rangle \oplus \mathbb{Z}\langle 4 - 4\sigma \rangle \oplus \mathbb{Z}\langle 10 - 4\lambda - 2\sigma \rangle$
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- This is very useful in the computation of $K(h)$-local Picard groups

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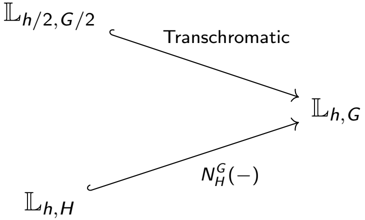

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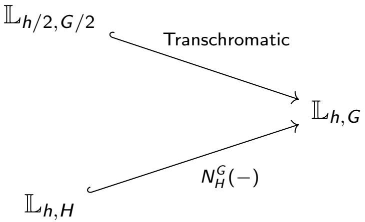

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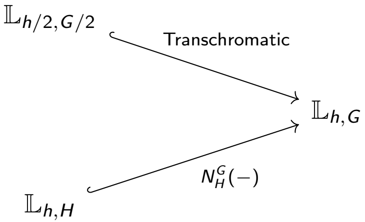

- The Transchromatic Isomorphism Theorem shows that a part of $\mathbb{L}_{h,G}$ is from $\mathbb{L}_{h/2,G/C_2}$
- The norm functor shows that if $V \in \mathbb{L}_{h,H}$, then $\mathrm{Ind}_H^G(V) \in \mathbb{L}_{h,G}$

Theorem (Duan–Hill–Li–Liu–S.–Wang–Xu)

1. *When $h = 2^{n-1}m$, the following are $RO(C_{2^n})$-periodicities of E_h:*
   - $1 + \sigma_{2^n} + 2^{n-2}\lambda_{n-1} + 2^{n-3}\lambda_{n-2} + \cdots + 2\lambda_2 + \lambda_1$
   - $2^{m+1} - 2^{m+1}\sigma_{2^n}$
   - $2^{2^{n-i}m+n-i+1} - 2^{2^{n-i}m+n-i}\lambda_{n-i}$, $1 \leq i \leq n-1$
2. *When $h = 4k+2$, the following are $RO(Q_8)$-periodicities of E_h:*
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- $\mathbb{L}^N_{h,G}$ is a pretty good approximation to $\mathbb{L}_{h,G}$
- In fact, $\mathbb{L}^N_{h,G}$ is a full rank sublattice of $RO(G)$

Full rank sublattice

Theorem (Duan–Hill–Li–Liu–S.–Wang–Xu)

1. *When* $(h, G) = (2^{n-1}m, C_{2^n})$,

$$RO(G)/\mathbb{L}^N_{h,G} \cong \bigoplus_{i=1}^{n-1} \mathbb{Z}/2^{2^{n-i-1}m+n-i} \oplus \mathbb{Z}/2^{h+n+1}$$

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Corollary

The complexity of $\pi_\star^G E_h$ *is finite, with a specific bound (given above).*

Happy birthday Peter!