The last Kervaire invariant

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The generalized Poincaré conjecture

Poincaré conjecture

A simply connected closed 3-manifold is the standard 3-sphere.

Generalized Poincaré conjecture

- topological case:
  Are all homotopy spheres homeomorphic to the standard one?
- smooth case:
  Are all n-dimensional homotopy spheres diffeomorphic to the standard one?

Generalized Poincaré conjecture

dimension	topological	smooth
≤ 2	true	true
3	true (Perelman)	true (Moise + Perelman)
4	true (Freedman)	unknown
≥ 5	true (Smale)	depends on dimension

Milnor's exotic 7-sphere

Milnor constructed a compact 8-manifold W with boundary as a disk bundle over the 4-sphere, such that:

- ∂W is a homotopy sphere
- $W \cup_{S^7} D^8$ does not admit any smooth structure extending the smooth structure on W:

  Pontryagin class not integral using Hirzebruch's signature formula.

Consequently, ∂W is not diffeomorphic to the standard sphere.

Kervaire's exotic 9-sphere

Kervaire constructed a compact 10-manifold M^{10} as follows:

1. Let U be the disk bundle associated with the tangent bundle of S^5
2. Let V be an embedded D^5 in S^5
3. Let M' be glued from two copies of U by identifying the two copies of $V \times D^5$ under the isomorphism $V \times D^5 \cong D^5 \times V$
4. M^{10} is obtained from M' by smoothing corners

Kervaire's exotic 9-sphere

Kervaire defined an invariant and showed that:

- ∂M^{10} is a homotopy sphere
- $M^{10} \cup_{S^9} D^{10}$ has Kervaire invariant 1
- any smooth 10-manifold has Kervaire 0

Consequetly,

- $M^{10} \cup_{S^9} D^{10}$ does not admit any smooth structure.
- ∂M^{10} is not diffeomorphic to the standard sphere

Kervaire invariant

Let $\Omega = \Omega S^6$, and M be a 4-connected closed 10-manifold.

- $H^5(\Omega) \cong \mathbb{Z}e_1$, $H^{10}(\Omega) \cong \mathbb{Z}e_2$
- for any $x \in H^5(M)$, there exits $f : M \to \Omega$ such that $f^*(e_1) = x$
- define $\Phi(x) \in \mathbb{Z}/2$ to be the mod 2 reduction of $f^*(e_2)$
- $\Phi : H^5(M, \mathbb{Z}/2) \to \mathbb{Z}/2$ is well-defined
- Φ is quadratic: $\Phi(x + y) = \Phi(x) + \Phi(y) + x \cup y$
- the Kervaire invariant of M is the Arf invariant of Φ (the majority of its value)

The construction can be generalized to any dimension $4k + 2$.

Classification of the group Θ_n of homotopy spheres, $n \geq 5$

(Kervaire-Milnor 1963)

- if $n \neq 4k+2$, there is an exact sequence

$$0 \to \Theta_n^{bp} \to \Theta_n \to \pi_n/J \to 0$$

- if $n = 4k+2$, there is an exact sequence

$$0 \to \Theta_n^{bp} \to \Theta_n \to \pi_n/J \xrightarrow{\Phi} \mathbb{Z}/2 \to \Theta_{n-1}^{bp} \to 0$$

- if n is even, $\Theta_n^{bp} = 0$.
- if $n = 4k$, Θ_{n-1}^{bp} is cyclic of order $2^{2k-2}(2^{2k-1}-1)c_k$, with c_k the numerator of $\frac{4B_{2k}}{k}$

Questions after Kervaire-Milnor

- How to compute the stable homotopy groups of spheres?
- When is the Kervaire invariant trivial?
- What is the story for $n = 4$?

The Kervaire invariant one problem

Does there exists framed n-dimensional smooth closed manifolds with non-trivial Kervaire invariant?

The Kervaire invariant one problem

Does there exists framed n-dimensional smooth closed manifolds with non-trivial Kervaire invariant?

In dimensions $2, 6, 14$, we can construct Kervaire invariant one framings on $S^1 \times S^1$, $S^3 \times S^3$, $S^7 \times S^7$ using trivializtions of the tangent bundles of S^1, S^3, S^7 respectively.

Kervaire invariant in the Adams spectral sequence

(Browder)

The Kervaire invariant is detected by h_i^2 in the Adams spectral sequnce.

- The Kervaire invariant is trivial if $n \neq 2^k - 2$
- The Kervaire invariant is detected by $\beta_{2^n,2^n}$ in the Adams-Novikov spectral sequence.

Construction of Kervaire classes

(Barratt, Jones, Mahowald, Tangora)

The Kervaire invariant is non-trivial in dimensions $30, 62$

Construction of Kervaire classes

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The Kervaire invariant is non-trivial in dimensions $30, 62$

They constructed the Kervaire class by exibiting its factorization through certain finite CW complexes.

$$S^{30} \to X \to S^0$$

$$S^{62} \to Y \to S^0$$

Non-existence of Kervaire classes

(Hill-Hopkins-Ravenel)

The Kervaire invariant is trivial in dimension ≥ 254

Non-existence of Kervaire classes

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The Kervaire invariant is trivial in dimension ≥ 254

They showed that $\beta_{2^n,2^n}$ cannot be a permanent cycle by comparing with the homotopy fixed points spectral sequence and the slice spectral sequence of certain C_8-equivariant spectrum.

$$ANSS(S^0) \to HFSS(E_4^{C_8}) \Rightarrow \pi_*(E_4^{hC_8})$$

$$SliceSS(E_4^{C_8}) \Rightarrow \pi_*(E_4^{C_8}) \to \pi_*(E_4^{hC_8})$$

The existence of θ_6

(Lin-Wang-Xu)

The Kervaire invariant in dimension 126 is non-trivial.

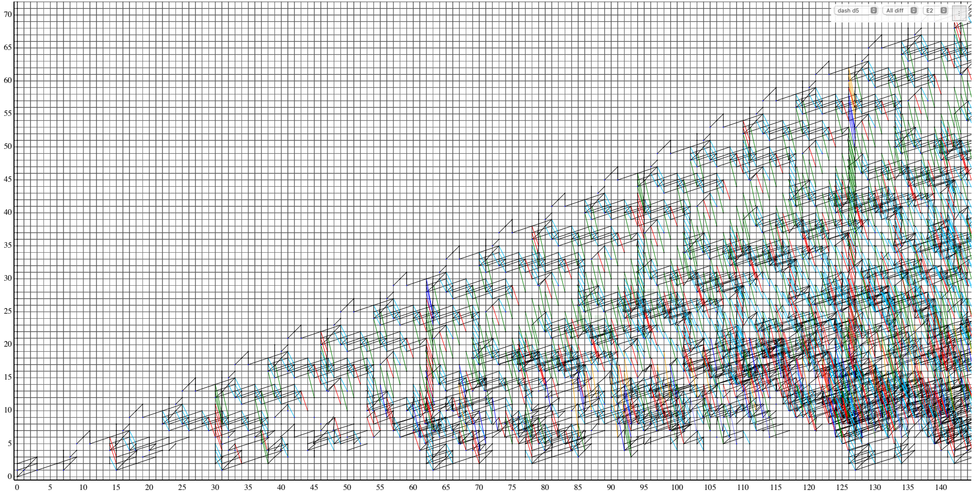
- h_6^2 is a permanent cycle in the Adans spectral sequence.
- half of the framed cobordism classes in dimension 126 does not contain homotopy spheres
- any homotopy sphere of dimension 125 bounding a framed manifold is diffeomorphic to the standard sphere

Remark: there exists exotic spheres in dimension 125 whose framed cobordism class is non-trivial.

Method of computation

- Adams spectral sequence:
  start from homological algebra over the Steenrod algerba
- generalized Adams spectral sequence:
  Adams-Novikov spectral sequence, motivic Adams spectral sequence
- deformation methods:
  motivic deformation, synthetic deformation
- Leibnitz rule:
  produce new differentials using multiplicative structure
- higher structures:
  Toda brackets, power operation, secondary operations
- Mahowald trick:
  dichotomy between Adams differential and multiplicative structure

Adams spectral sequence at $p = 2$ (Serre, Toda, May, Barratt, Mahowald, Tangora, Isaksen, Wang, Xu, Lin, ...)



Applications in the smooth Poincaré conjecture

Smooth Poincaré conjecture

Are all n-dimensional homotopy spheres diffeomorphic to the standard one?

- (Riemann-Roch, Moise) spheres of dimension 1,2,3 has unique smooth structure
- (Kervaire-Milnor, Isaksen, Wang-Xu) the spheres in dimension $5, 6, 12, 56, 61$ has unique smooth structure
- there exits exotic spheres in any other odd dimensions.
- (Behrens-Hill-Hopkins-Mahowald) there exits exotic spheres in other even dimensions from 5 to 138.
- (Behrens-Hill-Hopkins-Mahowald) The only dimensions up to 200 which we do not know if exotic spheres exist are $4, 140, 166, 176, 188$
- (Lin-Wang-Xu) there exits exotic spheres in dimensions $140, 166, 188$.

Main methods

- Use synthetic notions to make precise statements of extensions.
- Implement the generalized Leibniz rule on a computer.
- Construct a data base of Adams spectral sequences.

Synthetic spectra

Pstrągowski constructed the category of synthetic spectra

- symmetric monoidal stable ∞-category *synSp*
- symmetric monoidal functor $\nu : Sp \to synSp$
- $\lambda \in \pi_{0,-1}\nu\mathbb{S}$
- $\pi_{*,*}\nu(H\mathbb{F}_2) \cong \mathbb{F}_2[\lambda]$
- $\pi_{*,*}(\nu X/\lambda) \cong E_2^{*,*}(X)$
- $\lambda\text{-}BocSS(\nu X) \cong ASS(X)$

Synthetic extension

- $f : \Sigma^{0,j}\nu X \to \nu Y$
- $x \in E_2^{s,t}(X)$ survives to the E_r-page
- $y \in E_2^{s+j+n,t+j+n}(Y)$ survives to the E_{r-n}-page
- $f_r : \Sigma^{0,j}\nu X/\lambda^{r-1} \to \nu Y/\lambda^{r-1}$

We say that there is an f-extension on the E_r page (with jump of filtration n), if there is a synthetic f-extension

$$f_r(\tilde{x}) = \lambda^n \tilde{y}$$

for any lift $\tilde{x} \in \pi_{*,*}(\nu X)$, $\tilde{y} \in \pi_{*,*}(\nu Y)$ of x and y respectively. We denote it by

$$d_{n+j}^{f,E_r}(x) = y$$

Crossing extensions

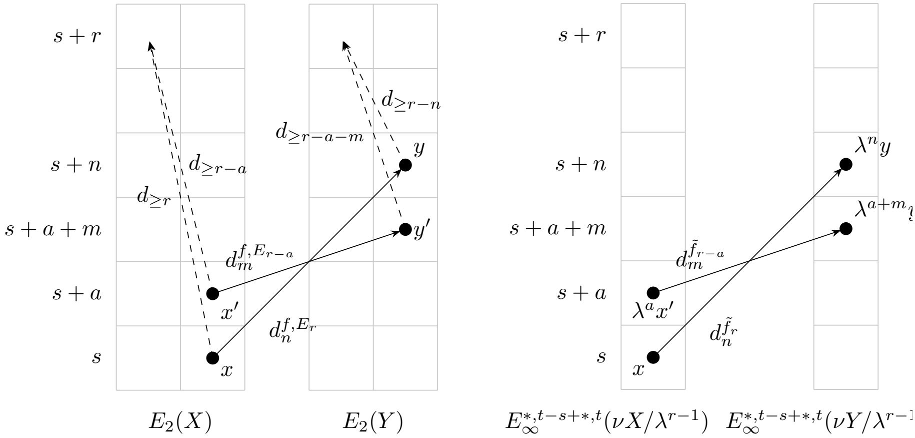
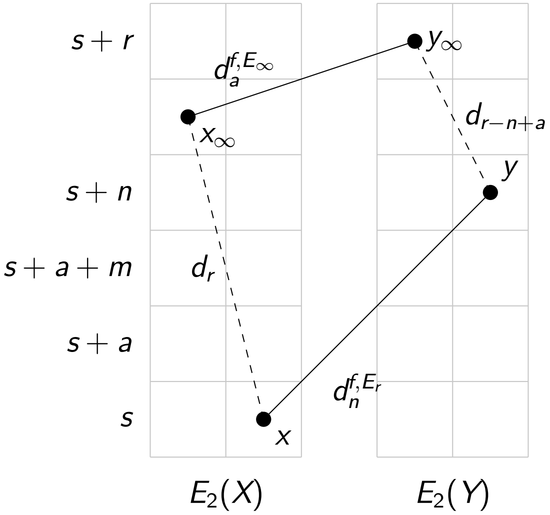


FIGURE 4. A crossing of an (f, E_r)-extension when $\mathrm{AF}(f) = 0$

Generalized Leibniz rule

1. $f: X \to Y$
2. $d_r(x) = x_\infty$
3. $d_n^{f,E_r}(x) = y$ for some $n \leq r-2$
4. $d_a^{f,E_\infty}(x_\infty) = y_\infty$ for some a

If either (2) or (3) has no crossing on the E_r-page, and (4) has no crossing on the E_∞-page, then y supports an Adams differential to y_∞



$s+r$
$s+n$
$s+a+m$
$s+a$
s
d_a^{f,E_∞}
y_∞
x_∞
d_{r-n+a}
y
d_r
d_n^{f,E_r}
x
$E_2(X)$
$E_2(Y)$

Genaralized Mahowald trick

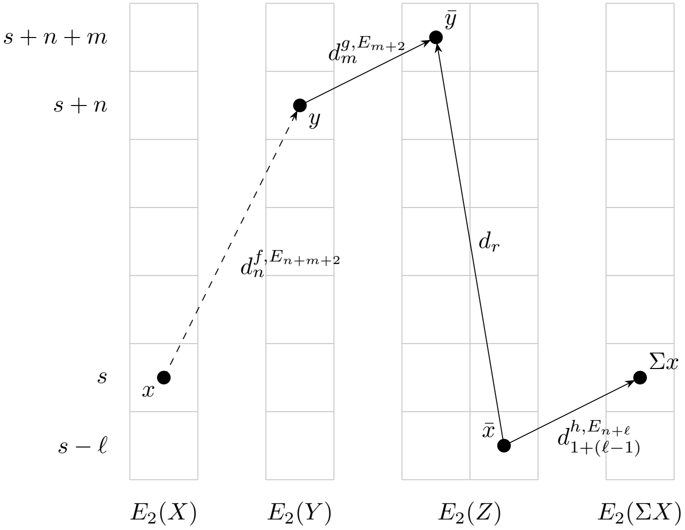
1. cofiber sequence $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$
2. $x \in E_2^{s,t}(X)$, $y \in E_2^{s+n,y+n}(Y)$
3. $\bar{x} \in E_2^{s-l,t-l+1}(Z)$ such that $d_l^{h,E_{r-m}}\bar{x} = \Sigma x$
4. $d_m^{g,E_{m+2}} y = \bar{y} \in E_2^{s-l+r,t-l+r}(Z)$
5. $d_r \bar{x} = \bar{y}$

If either (3) or (5) has no crossing, then

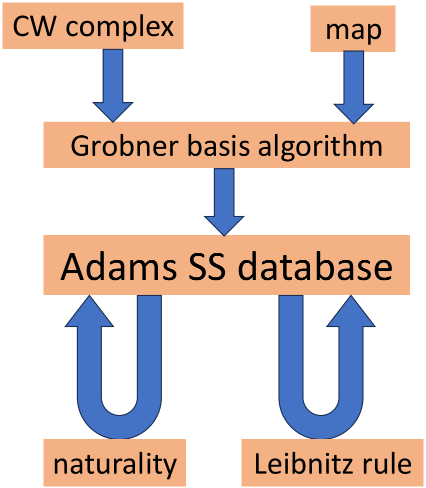
$$d_n^{f,E_{n+m+2}} x = y$$

modulo the image of $d_2, \ldots, d_{r-m}$ in the Adams spectral sequence of Y.

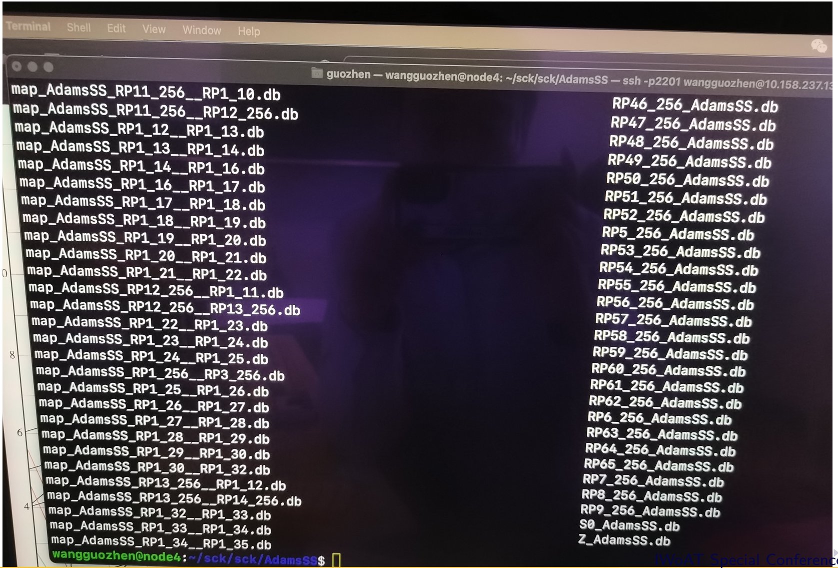
Generalized Mahowald trick



Implementation of computational techniques



The database of Adams spectral sequence



Thanks!