Proof for the existence of θ_6

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joint work with Weinan Lin and Guozhen Wang

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- dim 62 and 126, no explicit manifold known

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  - θ_6 exists $\Leftrightarrow \lambda\eta\theta_5^2 = 0$

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Theorem (Barratt–Jones–Mahowald, Burklund–Xu)

1. *The element h_6^2 survives to the E_{r+3}-page of the classical Adams spectral sequence if and only if for some θ_5,*

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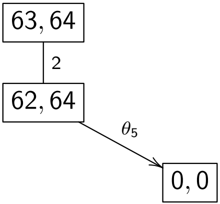

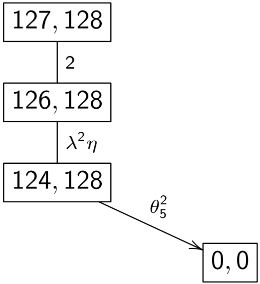

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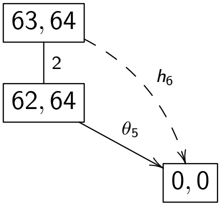

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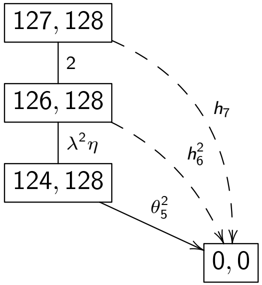

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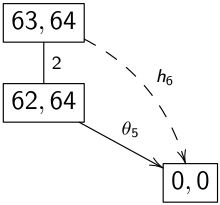

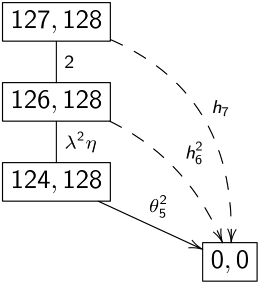

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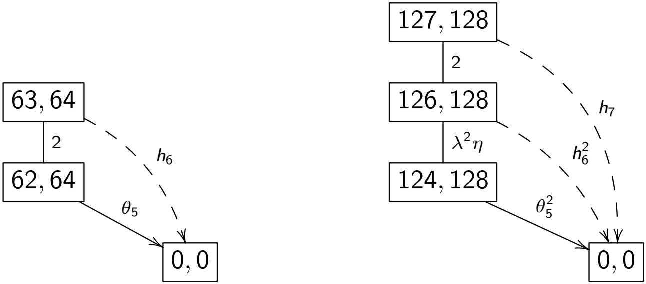

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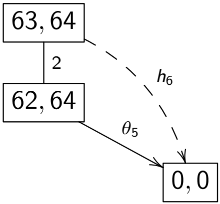

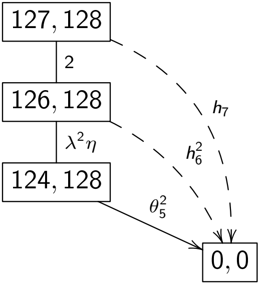

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Strategy for proving h_6^2 as a permanent cycle

- If $\eta\theta_5^2$ is detected by $\lambda^{n-5} T_n$ for some $T_n \in \mathrm{Ext}_A^{n,125+n}$, then there is a synthetic Adams differential $d_{n-2}(h_6^2) = \lambda^{n-3} T_n$.
- Goal: Show that $\lambda\eta\theta_5^2 = 0$ by Adams filtration (AF) estimation.
- Start with $\theta_5^2 \in \pi_{124,124+4} S^{0,0}$.
  - $\mathrm{Ext}_A^{i,125+i} = 0$ for $i \leqslant 2$
  - $\Rightarrow \pi_{124,124+4} S^{0,0}$ doesn't contain λ-torision.
- By inspection, $\mathrm{AF}(\theta_5^2) \geqslant 10$.
  - If $\mathrm{AF}(\theta_5^2) = 10$, then θ_5^2 is detected by $\lambda^6 h_0^2 x_{124,8}$.
- Next estimate $\eta\theta_5^2$.
  - $\mathrm{AF}(\lambda^3\eta[h_0^2 x_{124,8}]) \geqslant 14$.
  - If $\mathrm{AF}(\lambda^3\eta[h_0^2 x_{124,8}]) = 14$, then it is detected by $\lambda^6 h_1 h_4 x_{109,12}$.
  - If $\mathrm{AF}(\lambda^3\eta[h_0^2 x_{124,8}]) > 14$, then it is zero.

An η-extension

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Exactly one of (1) and (2) is true:

(1) h_6^2 *survives to the* E_∞*-page.*

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(3) $d_6(x_{126,8,4} + x_{126,8}) = 0.$

(4) $\theta_5^2 = \lambda^6[h_0^2 x_{124,8}] \neq 0 \in \pi_{124,124+4} S^{0,0}.$

(5) $\lambda^3 \eta [h_0^2 x_{124,8}] = \lambda^6 [h_1 h_4 x_{109,12}] \in \pi_{125,125+8} S^{0,0}.$

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Proposition B

If (3) is true, then (5) is not true.

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For the sake of contradiction, we assume (3) and (5) are both true.

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Lemma 2

The synthetic Toda bracket

$$\langle \lambda^3 \alpha_1, [h_0], \eta \rangle \subset \pi_{125,125+7} S^{0,0}/\lambda^9$$

does not contain zero, and is detected by $\lambda^4 h_0^2 x_{125,9,2}$.

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Lemma 3 (Corollary of Lemma 2)

$[\lambda^4 h_0^2 x_{125,9,2}] \cdot [h_0] = \lambda^6 [h_1 h_4 x_{109,12}] \neq 0 \in \pi_{125,125+8} S^{0,0}/\lambda^9.$

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- $\lambda\eta\alpha_1 + \lambda[h_0^2 x_{124,8}] \in \pi_{124,124+9}S^{0,0}/\lambda^9$ lies in AF $\geqslant 11$.
- $\lambda^3\eta\alpha_1 + \lambda^3[h_0^2 x_{124,8}] \in \pi_{124,124+7}S^{0,0}/\lambda^9$ has AF $\geqslant 13$ by inspection,

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1. $\lambda^3\eta \cdot \alpha_1 = \lambda^3[h_0^2 x_{124,8}] + \lambda^6\alpha_2 \quad \in \pi_{124,124+7}S^{0,0}/\lambda^9,$
   $\eta \cdot \alpha_2 = \lambda \cdot \alpha_3 \quad \in \pi_{125,125+14}S^{0,0}/\lambda^9,$
2. $\lambda^3 \cdot \alpha_1 \cdot [h_0] = 0 \quad \in \pi_{123,123+7}S^{0,0}/\lambda^9.$

Facts: (1) $x_{123,9} + h_0 x_{123,8}$ survives to E_{12}, not killed by any differential.
(2) $d_2(x_{125,8}) = h_1(x_{123,9} + h_0 x_{123,8}) + h_0^2 x_{124,8}$.

- Let $\alpha_1 = [x_{123,9} + h_0 x_{123,8}] \in \pi_{123,123+9}S^{0,0}/\lambda^{11}$ and its images in $\pi_{123,123+9}S^{0,0}/\lambda^r$ for $1 \leqslant r \leqslant 10$.
- $\lambda\eta\alpha_1 + \lambda[h_0^2 x_{124,8}] \in \pi_{124,124+9}S^{0,0}/\lambda^9$ lies in AF $\geqslant 11$.
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Sketch Proof of Lemmas 2 and 3

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The synthetic Toda bracket $\langle \lambda^3 \alpha_1, [h_0], \eta \rangle \subset \pi_{125,125+7} S^{0,0}/\lambda^9$ *does not contain zero, and is detected by* $\lambda^4 h_0^2 x_{125,9,2}$.

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Hopf, Kervaire, and $\cdots$

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  - $\mathrm{Ext}_A^{3,*}$: other than h_j^3, many cases remain

$$h_j^2 h_{j+k+1} + h_{j+1} h_{j+k}^2 = \langle h_j^2, \ h_0, \ h_{j+k}^2 \rangle.$$

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$$Sq^0 h_j = h_{j+1},\ Sq^0 h_j^2 = h_{j+1}^2,\ Sq^0 h_j^3 = h_{j+1}^3$$

- Sq^0-family: $x,\ Sq^0 x,\ Sq^0(Sq^0 x),\ \cdots$
- New Doomsday Conjecture: For any nonzero Sq^0-family, only finitely many classes survive.
  - $\mathrm{Ext}_A^{1,*} \Leftrightarrow$ Hopf invariant problem,
  - $\mathrm{Ext}_A^{2,*} \Leftrightarrow$ Kervaire invariant problem,
  - $\mathrm{Ext}_A^{3,*}$: other than h_j^3, many cases remain

$$h_j^2 h_{j+k+1} + h_{j+1} h_{j+k}^2 = \langle h_j^2,\ h_0,\ h_{j+k}^2 \rangle.$$

- Uniform Doomsday Conjecture: For any nonzero Sq^0-family $\{a_j\}$, there exists a Sq^0-family $\{b_j\}$, $r \geqslant 2$, $c \in \mathrm{Ext}$, such that

$$d_r(a_j) = c \cdot b_j \neq 0, \text{ for } j >> 0.$$

Thank you!