Problems on Units and Thom Spectra

IWoAT Summer School 2026

Stable Homotopy Theory

Problems to accompany the lecture

The following problems are intended to provide guidance on verifying claims in the lecture and ensuring understanding of the content.

Problem 1. Transition functions and structure groups.

Let $p : E \to X$ be a rank-n real vector bundle, and let

$$\varphi_i : p^{-1}(U_i) \xrightarrow{\cong} U_i \times \mathbb{R}^n$$

be local trivializations.

(a) Show that on an overlap $U_i \cap U_j$, the change of trivialization is of the form

$$\varphi_i \varphi_j^{-1}(x, v) = (x, g_{ij}(x)v)$$

for a continuous map

$$g_{ij} : U_i \cap U_j \longrightarrow \mathrm{GL}_n(\mathbb{R}).$$

(b) Verify the cocycle condition

$$g_{ij}(x) g_{jk}(x) = g_{ik}(x)$$

on triple overlaps.

(c) After choosing a fiberwise inner product on E, explain why the structure group may be reduced from $\mathrm{GL}_n(\mathbb{R})$ to $O(n)$.

(d) Interpret the classifying map

$$c_E : X \longrightarrow BO(n)$$

in terms of this transition data.

Problem 2. From vector bundles to spherical fibrations and the J-map.

Let $E \to X$ be a rank-n real vector bundle.

(a) Show that a linear isomorphism

$$A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

extends to a based homeomorphism of one-point compactifications

$$A^+ : S^n \longrightarrow S^n.$$

(b) Deduce that fiberwise one-point compactification sends the transition functions of E to transition functions for a based S^n - bundle.

(c) If

$$J_n : BO(n) \longrightarrow B\,\mathrm{hAut}_*(S^n)$$

denotes the resulting classifying map, explain why the spherical fibration associated to E is classified by

$$X \xrightarrow{c_E} BO(n) \xrightarrow{J_n} B\,\mathrm{hAut}_*(S^n).$$

(d) Explain how stabilization leads to the stable J - homomorphism

$$J : BO \longrightarrow B\,\mathrm{GL}_1(\mathbb{S}).$$

In this language, what should the notation $J(E)$ mean for a stable vector bundle E over X?

Problem 3. ***MO* as a Thom Spectrum.**

Let $E \to X$ be a rank-n vector bundle, and let

$$J(E) : X \longrightarrow B\,\mathrm{GL}_1(\mathbb{S})$$

be its stabilized spherical fibration. Give a conceptual argument for the equivalence

$$M(J(E)) \simeq \Sigma^{-n}\Sigma^{\infty}\,\mathrm{Th}(E).$$

Use this to explain why the Thom spectrum of the universal stable J - homomorphism

$$BO \xrightarrow{J} B\,\mathrm{GL}_1(\mathbb{S})$$

is MO.

Problem 4. The unit space and the unit spectrum.

Let R be an E_∞ - ring spectrum.

(a) Show that there is a natural equivalence of spaces

$$\mathrm{Map}_{R\text{-}\mathrm{mod}}(R, R) \simeq \Omega^\infty R.$$

(b) Show that an endomorphism of the free rank-one R-module R is an equivalence exactly when its class in $\pi_0 R$ is a unit. Deduce that

$$\mathrm{GL}_1(R) \simeq \coprod_{u \in (\pi_0 R)^\times} (\Omega^\infty R)_u.$$

(c) Since $\mathrm{GL}_1(R)$ is a grouplike $\mathbb{E}_\infty$-space, it determines a connective spectrum $gl_1(R)$ satisfying

$$\Omega^\infty gl_1(R) \simeq \mathrm{GL}_1(R).$$

Show that

$$\pi_k gl_1(R) \cong \begin{cases} (\pi_0 R)^\times, & k = 0, \\ \pi_k R, & k > 0. \end{cases}$$

(d) Compute $\pi_0 gl_1(\mathbb{S})$. What is $\mathrm{GL}_1(HA)$ when A is an ordinary commutative ring?

Problem 5. Why is a Thom spectrum an R-module?

Let

$$f : X \longrightarrow R\text{-line} \simeq B\,\mathrm{GL}_1(R)$$

classify an R-line bundle. Define

$$M(f) = \operatorname{colim}\left(X \xrightarrow{f} R\text{-line} \hookrightarrow R\text{-mod}\right),$$

where the colimit is taken in R-mod.

(a) Explain formally why $M(f)$ is an object of R-mod.

(b) Explain informally why this is reasonable from the viewpoint of "gluing copies of R by R-linear equivalences."

(c) Suppose that f is nullhomotopic. Show that

$$M(f) \simeq R \wedge \Sigma^\infty_+ X.$$

(d) Specialize to $R = \mathbb{S}$ and explain why this recovers the suspension spectrum of X_+ for the trivial $\mathbb{S}$-line bundle.

Problem 6. Multiplicativity of the Thom construction.

(a) For real vector bundles $E \to X$ and $F \to Y$, verify the classical formula

$$\mathrm{Th}(E \boxplus F) \simeq \mathrm{Th}(E) \wedge \mathrm{Th}(F),$$

where $E \boxplus F \to X \times Y$ is the external Whitney sum.

(b) If L and L' are R-lines, show that

$$L \otimes_R L'$$

is again an R-line-line. Thus R-line inherits a symmetric monoidal structure from R-mod.

(c) Explain why it is natural to expect a symmetric monoidal Thom functor to carry an $\mathbb{E}_k$-monoidal map into R-line to an $\mathbb{E}_k$-R-algebra.

(d) Apply this philosophy to

$$J : BO \longrightarrow B\,\mathrm{GL}_1(\mathbb{S})$$

and the Whitney-sum multiplication on BO. What multiplicative structure should one expect on MO?

Problem 7. Orientations as trivializations.

Let $R \to A$ be a map of $\mathbb{E}_\infty$-ring spectra, and let

$$f : X \longrightarrow R\text{-line} \simeq B\,\mathrm{GL}_1(R)$$

classify an R - line bundle. Extension of scalars sends an R - line L to the A - line

$$L \otimes_R A \simeq A,$$

and therefore induces a map

$$R\text{ - line} \longrightarrow A\text{ - line}.$$

Write

$$f_A : X \longrightarrow A\text{ - line}$$

for the composite.

(a) Show that extension of scalars commutes with the Thom construction:

$$M(f) \otimes_R A \simeq M(f_A).$$

(b) An A - orientation of f may be viewed as a trivialization of the A - line bundle f_A. Explain why such a trivialization is equivalent to a homotopy from f_A to the constant map at the distinguished A - line A.

(c) Suppose $R = \mathbb{S}$ and $f = J(E)$ for a real vector bundle $E \to X$. Explain how this formulation recovers the familiar idea that an orientation trivializes the twisting detected by the associated spherical fibration after passing to the chosen cohomology theory A.

Problems to apply the lecture

The following problems are intended to connect the material on units and Thom spectra with themes from the rest of the summer school.

Problem 1. Monodromy, homotopy colimits, and cofibers.

Let R be an $\mathbb{E}_\infty$ - ring spectrum and let

$$u \in (\pi_0 R)^\times.$$

The element u determines an R - line bundle

$$f_u : S^1 = B\mathbb{Z} \longrightarrow B\,\mathrm{GL}_1(R)$$

whose monodromy is multiplication by u.

(a) Regard f_u as a functor

$$B\mathbb{Z} \longrightarrow R\text{ - mod}$$

which sends the unique object to R and the generator $1 \in \mathbb{Z}$ to the R - module equivalence $u : R \to R$.

(b) Using a standard two-term resolution for the $\mathbb{Z}$-action, or an equivalent homotopy-colimit computation, show that there is a cofiber sequence

$$R \xrightarrow{u-1} R \longrightarrow M(f_u).$$

Hence

$$M(f_u) \simeq \mathrm{cofib}(u-1).$$

(c) Take $R = \mathbb{S}_p^\wedge$ and $u = 1 - p$. Identify $M(f_u)$.

(d) Explain which parts of the argument use stability of R-mod and which use the interpretation of a Thom spectrum as a homotopy colimit.

Problem 2. Change of rings, orientations, and Thom isomorphisms.

Let $R \to A$ be a map of $\mathbb{E}_\infty$-ring spectra and let

$$f : X \longrightarrow R\text{-line}$$

be an R-line bundle. Let

$$f_A : X \longrightarrow A\text{-line}$$

be obtained by extension of scalars.

(a) Recall that

$$- \otimes_R A : R\text{-mod} \longrightarrow A\text{-mod}$$

is a left adjoint. Use this to prove that it preserves colimits.

(b) Deduce directly from the colimit definition of the Thom spectrum that

$$M(f) \otimes_R A \simeq M(f_A).$$

(c) Suppose f_A is nullhomotopic. Show that

$$M(f) \otimes_R A \simeq A \wedge \Sigma^\infty_+ X.$$

(d) Explain why part (c) may be interpreted as a Thom isomorphism associated to an A-orientation of f.

(e) Suppose $R = \mathbb{S}$ and L is a smashing symmetric monoidal Bousfield localization. Taking $A = L\mathbb{S}$, explain why the preceding result identifies

$$L\, M(f) \simeq M(f_{L\mathbb{S}}),$$

where $f_{L\mathbb{S}}$ is obtained from f by extension of scalars to $L\mathbb{S}$.

Problem 3. Multiplicativity from preservation of colimits.

Let R be an E_∞-ring spectrum and let

$$f : X \longrightarrow R\text{-line}, \qquad g : Y \longrightarrow R\text{-line}$$

be R-line bundles. Define their external tensor product

$$f \boxtimes g : X \times Y \longrightarrow R\text{-line}$$

by

$$(f \boxtimes g)(x, y) = f(x) \otimes_R g(y).$$

(a) Show that $f(x) \otimes_R g(y)$ is again an R-line.

(b) Using that $\otimes_R$ preserves colimits separately in each variable, prove

$$M(f \boxtimes g) \simeq M(f) \otimes_R M(g).$$

(c) Let X carry an $\mathbb{E}_1$ - multiplication and suppose

$$f : X \longrightarrow R\text{ - line}$$

is compatible with this multiplication. Explain how part (b) produces an $\mathbb{E}_1$ - R - algebra structure on $M(f)$.

(d) What changes if the input is $\mathbb{E}_\infty$ rather than $\mathbb{E}_1$?