Units & Thom Spectra

IWoAT 2026

Unstable Thom Construction

Let $E \to X$ be an n-dim'l vector bundle

$$\begin{tikzcd} G\mathbb{R}^n \arrow[r] & E \arrow[d] \\ & X \end{tikzcd}$$

$\rightsquigarrow$ The transition data comes from elements of $\boxed{O(n)}$

Where does $O(n)$ come from?

Transition data = gluing instructions on local trivializations

= $\mathbb{R}$-linear automorphisms of $\mathbb{R}^n$

= elements of $\mathrm{Aut}_{\mathbb{R}}(\mathbb{R}^n) \cong GL_n(\mathbb{R})$

Unstable Thom Construction

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$$\begin{tikzcd} \circlearrowleft\mathbb{R}^n \arrow[r] & E \arrow[d] \\ & X \end{tikzcd}$$

We can reduce the structure group from $GL_n(\mathbb{R})$!

Unstable Thom Construction

Let $E \longrightarrow X$ be an n-dim'l vector bundle

$$\begin{array}{ccc} \mathbb{R}^n & \longrightarrow & E \\ & & \downarrow \\ & & X \end{array}$$

$\rightsquigarrow$ The transition data comes from elements of $O(n)$

$\rightsquigarrow$ The classifying space of n-dim'l vector bundles is $\boxed{BO(n)} \simeq BGL_n(\mathbb{R})$

classification space of automorphisms of the local model

Unstable Thom Construction

Let $E \to X$ be an n-dim'l vector bundle

$$\begin{tikzcd} \mathbb{R}^n \arrow[r] & E \arrow[d] \\ & X \end{tikzcd}$$

⇝ The transition data comes from elements of $O(n)$

⇝ The classifying space of n-dim'l vector bundles is $BO(n)$

$E \to X$ is classified by a map $X \xrightarrow{c_E} BO(n)$

$$\begin{tikzcd} E \cong c_E^*(\gamma_n) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \gamma_n \arrow[d] \\ X \arrow[r, "c_E"'] & BO(n) \end{tikzcd}$$

Unstable Thom Construction

The Thom space associated to $E \longrightarrow X$ is constructed as follows:

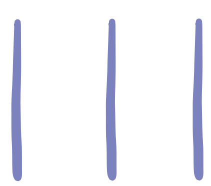
The vector bundle: $E \longrightarrow X$

↳ ① Take the fiberwise one-pt compactification

↳ The associated sphere bundle: $E^+ \longrightarrow X$

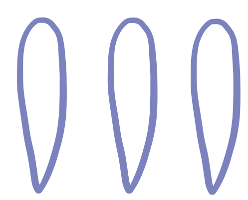
↳ ② Quotient by the section at ∞

↳ the Thom space: $Th(E)$



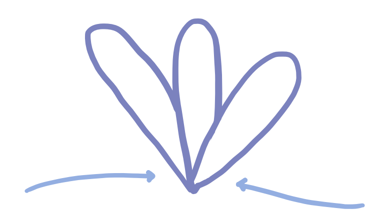
$$\begin{array}{c} \mathbb{R}^n \\ \downarrow \\ E \\ \downarrow \\ X \end{array}$$

①



$$\begin{tikzcd} S^n \arrow[d] \\ E^+ \arrow[d] \\ X \arrow[u, bend right, "s_\infty"'] \end{tikzcd}$$

②



$$E^+/s_\infty(X) =: Th(E)$$

Fact: For ε^n the trivial bundle, $Th(\varepsilon^n \oplus E) \cong S^n \wedge Th(E)$.

Unstable Thom Construction

The inclusion $BO(n) \hookrightarrow BO(n+1)$ classifies $\varepsilon^1 \oplus \gamma_n$

$\rightsquigarrow$ passing to Thom spaces yields

$$\underbrace{Th(\varepsilon^1 \oplus \gamma_n)}_{\substack{\cong \\ S^1 \wedge Th(\gamma_n) \\ \cong \\ \Sigma Th(\gamma_n)}} \xrightarrow{s_n} Th(\gamma_n)$$

Example.

unoriented cobordism: MO w/

levels $MO_n := Th(\gamma_n)$

& structure maps $s_n : \Sigma Th(\gamma_n) \longrightarrow Th(\gamma_n)$

Some Questions

① What happens when we stabilize?

② Is there a Thom construction for other fiber bundles?

③ " ... other bundle types?

④ Are there any nice properties?

⑤ Why should we care?

Stable Vector Bundles

Stable vector bundles are those where shifting the dimension to find equality

$\rightsquigarrow$ $\boxed{E, F}$ vector bundles s.t. for some n, m we have

Need NOT have the same dimension

$$\varepsilon^n \oplus E \cong \varepsilon^m \oplus F \quad \Rightarrow \quad E \simeq_s F$$

$\rightsquigarrow$ The transition data comes from some element of $O(k)$, where $\boxed{k} = n + |E| = m + |F|$

BUT this k is <u>not unique!</u>

$$\varepsilon^n \oplus E \cong \varepsilon^m \oplus F \Rightarrow \varepsilon^{n+1} \oplus E \cong \varepsilon^{m+1} \oplus F$$

Stable Vector Bundles

Stable vector bundles are those where shifting the dimension to find equality

$\rightsquigarrow$ E, F vector bundles s.t. for some n, m we have
$\varepsilon^n \oplus E \cong \varepsilon^m \oplus F \Rightarrow E \simeq_s F$

$\rightsquigarrow$ The transition data comes from some element of $O(k)$, where $k = n + |E| = m + |F|$

$\rightsquigarrow$ The transition data comes from an element of $O := \varinjlim O(k)$

$\rightsquigarrow$ The classifying space of stable vector bundles is BO

$E_s \to X$ is classified by a map $X \xrightarrow{c_{E_s}} BO$

By pulling back along $\gamma := \varinjlim \gamma_n$

Stable Vector Bundles

Our levelwise construction of MO depended on keeping track of an n-dim'l fiber ...

... but is there a description of MO that's intrinsically stable?

Well, what information did the Thom space retain about $E \to X$?

$$\begin{array}{c} E \\ \downarrow \\ X \end{array} \rightsquigarrow E^+/s_\infty(X)$$

The Thom space only sees the associated spherical bundle!

... so what if we just started with spherical bundles?

Spherical Bundles

Spherical bundles follow a similar story to vector bundles.

Let $F \to X$ be an n-sphere bundle

$$\begin{array}{ccc} S^n & \longrightarrow & F \\ & & \downarrow \\ & & X \end{array}$$

⇝ The transition data comes from elements of $hAut_*(S^n) =: G(n)$

⇝ The classifying space of n-sphere bundle is $BG(n)$

$F \to X$ is classified by a map $X \xrightarrow{c_F} BG(n)$

⇝ There is a map between classifying spaces

$$BO(n) \longrightarrow BG(n)$$

Stable Spherical Bundles

What becomes of the fibers after stabilizing a spherical bundle?

$$S^n \rightsquigarrow \Sigma^\infty S^n \rightsquigarrow \Sigma^{-n}\Sigma^\infty S^n \cong \mathbb{S}$$

Stable spherical bundle $\longleftrightarrow$ Family of objects locally equivalent to $\mathbb{S}$

Remark: Spc is the ∞-category of ∞-groupoids & Sp is the ∞-category of spectra

$\rightsquigarrow$ The transition data comes from elements of $\overset{GL_1(\mathbb{S})}{\cancel{G := \varinjlim}}\, G(n)$

transition data should come from $\boxed{\mathrm{Aut}_{\mathbb{S}}(\mathbb{S})}$

These are rank 1 automorphisms of $\mathbb{S}$-modules

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$\rightsquigarrow$ The transition data comes from elements of $\cancel{G := \varinjlim G(n)}$ $GL_1(\mathbb{S})$

$\rightsquigarrow$ The classifying space of stable spherical bundles is $BG \simeq BGL_1(\mathbb{S})$

$F_s \longrightarrow X$ is classified by a map $X \xrightarrow{c_{F_s}} BGL_1(\mathbb{S})$

$\mathbb{S}$-lines and the J-homomorphism

Let $\mathbb{S}$-line $:=$ full subgpd$_{Sp^{\simeq}}$ $\{L \in Sp : L \simeq \mathbb{S}\}$; ptd with bspt $\mathbb{S}$

$\rightsquigarrow$ $\mathbb{S}$-line is connected $\Rightarrow$ $\mathbb{S}\text{-line} \simeq B\,\boxed{\mathrm{Aut}_{\mathbb{S}}(\mathbb{S})}$

$\simeq GL_1(\mathbb{S})$

$\rightsquigarrow$ $\mathbb{S}\text{-line} \simeq BGL_1(\mathbb{S})$ $\rightsquigarrow$

stabilization didn't forget about the relationship between vector bundles & spherical bundles!

The J-homomorphism is $BO \longrightarrow BGL_1(\mathbb{S}) \simeq \mathbb{S}\text{-line}$

stable vector bundle $\longmapsto$ associated stable spherical bundle

much like the Thom space!

by forgetting the vector space structure, but remembering the stable spherical fibers

Stable Thom construction

Let $E \to X$ be a vector bundle

$\rightsquigarrow$ The associated J-map is $X \xrightarrow{J(E)} S\text{-line}$ is given by taking the appropriate precompositions with J.

$\rightsquigarrow$ NOTE: We can describe E in terms of it's fibers

$$E = \operatorname{colim}_x E_x$$

where the colimit is taken with respect to transition data.

$\rightsquigarrow$ The associated Thom spectrum to E is $\lfloor \operatorname{colim}_x (J(E))_x \rceil$

also written $\operatorname{colim}_X (J(E))$

Really we're taking the colimit of

$$X \xrightarrow[J(E)]{} S\text{-line} \hookrightarrow Sp$$

Stable Thom construction: MO

Example.

Take $\gamma_n \longrightarrow BO(n)$ $\rightsquigarrow$ $J(\gamma_n) : BO(n) \longrightarrow S\text{-line}$

$\rightsquigarrow \operatorname*{colim}_{BO(n)} \left(J(\gamma_n) \right) \simeq \Sigma^{-n} \Sigma^{\infty} Th(\gamma_n)$

Take $\gamma \longrightarrow BO$ $\rightsquigarrow$ $J(\gamma) : BO \longrightarrow S\text{-line}$

$$\rightsquigarrow \operatorname*{colim}_{BO} \left(J(\gamma) \right) \simeq \varinjlim \operatorname*{colim}_{BO(n)} \left(J(\gamma_n) \right) \simeq \underbrace{\varinjlim \Sigma^{-n} \Sigma^{\infty} Th(\gamma_n)}_{\simeq MO \text{ !!}}$$

So our intrinsically stable presentation of MO?

$\rightsquigarrow \operatorname*{colim}_{BO} \left(BO \longrightarrow S\text{-line} \right) \rightsquigarrow$

R-lines

(R a ring spectrum)

let **R-line** := full subgpd$_{R\text{-mod}^{\simeq}}$ $\{M \in R\text{-mod} : M \simeq R\}$; ptd with bspt R

⇝ An R-module M is **free of rank 1** if $\exists\, M \xrightarrow{\sim} R$ in R-mod

⇝ R-line is connected ⇒ $S\text{-line} \simeq B\mathrm{Aut}_R(R)$

$\hookrightarrow \simeq GL_1(R)$

⇝ $R\text{-line} \simeq BGL_1(R)$ ⇜

For $X \in \mathrm{Spc}$, an **R-line bundle** on X is a map $X \xrightarrow{f} R\text{-line}$.

⇝ The **associated Thom spectrum** is

$$Mf := \mathrm{colim}\{X \longrightarrow R\text{-line} \hookrightarrow R\text{-mod}\}$$

REMARK: Mf is a spectrum, but Mf is also an $\boxed{\text{R-module}}$.

Each fiber is $R \in R\text{-mod}$ & the colimit is gluing the fibers via R-linear maps.

Unit Space, Unit Spectrum

The unit space is $GL_1(R) \simeq \coprod_{u \in (\pi_0 R)^\times} (\Omega^\infty R)_u$

$\rightsquigarrow$ $\pi_k GL_1(R) = \pi_k(R)$ for $k > 0$

$\rightsquigarrow$ $\pi_0 GL_1(R) = (\pi_0 R)^\times$ $\Rightarrow$ $GL_1(R)$ is group like

$\rightsquigarrow$ The structure on R, implies the structure on $GL_1(R)$

↳ R E_n-algebra $\rightsquigarrow$ $GL_1(R)$ E_n-space

↳ R E_∞-algebra $\rightsquigarrow$ $GL_1(R)$ E_∞-space

$\rightsquigarrow$ Following from the Recognition principle:

$\exists$ some connective spectrum, $gl_1(R)$, s.t

$$\Omega^\infty gl_1(R) \simeq GL_1(R)$$

Unit Space, Unit Spectrum

The unit spectrum is precisely this connective spectrum $gl_1(R)$.

$gl_1(R)$ packages the units into a genuinely stable object!

⇝ The unit spectrum is not $R_{\geq 0}$!

	$\pi_k \, gl_1(R)$	$\pi_k R$
$k<0$	0	$\pi_k R$
$k=0$	$(\pi_0 R)^\times$	$\pi_0 R$
$k>0$	$\pi_k R$	$\pi_k R$

⇝ There's an adjunction

$$\Sigma^\infty_+ \Omega^\infty : Sp_{\geq 0} \rightleftarrows CAlg(Sp) : gl_1$$

Assembly Map

$\boxed{\Sigma^\infty_+ \Omega^\infty : Sp_{\geq 0} \dashv CAlg(Sp) : gl_1}$ is reminiscent of

$\mathbb{Z}[-] : Gp \dashv Ring : (-)^\times$

$\ulcorner Map_{CAlg}(\Sigma^\infty_+ \Omega^\infty E, R) \cong Map_{Sp_{\geq 0}}(E, gl_1(R)) \urcorner$

This is how the Thom construction turns geometric twisting data into an R-module.

Assembly Map

$\Sigma^\infty_+ \Omega^\infty : Sp_{\geq 0} \dashv CAlg(Sp) : gl_1$

is reminiscent of

$$\boxed{\mathbb{Z}[-] : Gp \dashv Ring : (-)^\times}$$

The counit here defines an assembly map

$$\mathbb{Z}[R^\times] \longrightarrow R$$

To understand the structure of a ring, we can strip it down to units and look at how to "assemble" our pieces

Assembly Map

$\Sigma^\infty_+ \Omega^\infty : Sp_{\geq 0} \dashv CAlg(Sp) : gl_1$

is reminiscent of

$\mathbb{Z}[-] : Gp \dashv Ring : (-)^\times$

$\rightsquigarrow$ The counit of the adjunction $\Sigma^\infty_+ \Omega^\infty \dashv gl_1$ is

$$\underbrace{\Sigma^\infty_+ \Omega^\infty gl_1(R)}_{\substack{\cong \\ GL_1(R)}} \longrightarrow R$$

The spectral assembly map is the counit above, or equivalently

$$\underbrace{\Sigma^\infty_+ GL_1(R) \longrightarrow R}_{\text{"}GL_1(R) \text{ acts on the rank 1 module } R\text{"}}$$

Functoriality

Recall Mf is itself an R-module.

Construction of the Thom spectrum yields a colimit preserving functor

$$M : \mathrm{Spc}_{/R\text{-line}} \longrightarrow R\text{-mod}$$

$\rightsquigarrow$ Take $X \xrightarrow{f} R\text{-line}$, $Y \xrightarrow{g} R\text{-line}$, & $\boxed{h : f \to g}$

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & \arrow[d, phantom, "////"] & Y \arrow[dl, "g"] \\ & R\text{-line} & \end{tikzcd}$$

$$\rightsquigarrow \quad Mg = \operatorname{colim}_y g(y)$$

& for each y $\exists$ canonical $g(y) \longrightarrow Mg$

for each $x \in X$: $f(x) \simeq g(\boxed{h(x)}) \longrightarrow Mg$

$$\begin{tikzcd} \{f(x)\}_{x \in X} \arrow[rr] \arrow[dr] & & Mg \\ & Mf \arrow[ur] & \end{tikzcd}$$

$\Rightarrow$ $\{f(x)\}_{x \in X} \longrightarrow Mg$, but $\{f(x)\}_{x \in X} \longrightarrow Mf$ is universal!

Maps of twisting data $\rightsquigarrow$ Maps of Thom spectra!

Functoriality: Multiplicativity

Fact about Thom spaces: $Th(E \boxplus F) \simeq Th(E) \wedge Th(F)$,
where $\boxplus$ is the external Whitney sum

because of the colimit structure

sum of bundles $\longmapsto$ smash of Thoms

Take $L, K \in R\text{-line}$ $(L \simeq R \simeq K)$,
then $L \otimes_R K \simeq R \otimes_R R \simeq R$,
so $L \otimes_R K \in R\text{-line}$.

$\Rightarrow$ $\otimes$ on R-mod restricts to R-line!

$\rightsquigarrow$ $Spc_{/R\text{-line}}$ has a monoidal structure!

$$f \boxtimes g : X \times Y \xrightarrow{f \times g} R\text{-line} \times R\text{-line} \xrightarrow{\otimes_R} R\text{-line}$$

$\rightsquigarrow$ $M(f \boxtimes g) \simeq Mf \otimes_R Mg$ falls right out

E_n-structure on twisting data $\Rightarrow$ E_n-structure on Thom spectrum

∞ vs. Model Categorically

§6.4 constructs point set models that give the rigidity necessary to support actual structured ring spectra!

$\mathbb{L}$-spaces:

$f: X \longrightarrow B_{\times_{\mathbb{L}}} \boxed{GL_1^c(R)}$ classifies some principal $GL_1^c(R)$-object

actually a cofibrant replacement.

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E_{\times_{\mathbb{L}}} GL_1^c(R) \arrow[d] \\ X \arrow[r] & B_{\times_{\mathbb{L}}} GL_1^c(R) \end{tikzcd}$$

then we have $\boxed{GL_1^c(R) \circlearrowright P^c \ \& \ GL_1^c(R) \circlearrowright R}$

In ordinary geometry $P^c \times_{GL_1^c(R)} R$

& so we can define $\underbrace{(\Sigma^\infty_{\mathbb{L}})_+ P^c \wedge_{(\Sigma^\infty_{\mathbb{L}})_+ GL_1^c(R)} R}_{\text{the spectral derived analogue}}$

I-spaces: has convenient multiplicative properties

Used to show the Thom functor is lax symmetric monoidal.

∞ vs. Model Categorically

∞-categorical	Model Categorical
$BGL_1(R)$	$B_{\times_{\mathbb{L}}} GL_1^c(R)$
$\Sigma^\infty_+ GL_1(R) \longrightarrow R$	$(\Sigma^\infty_{\mathbb{L}})_+ GL_1^c(R) \longrightarrow R$
$Mf = \operatorname{colim}(\boxed{i} \circ f)$	$(\Sigma^\infty_{\mathbb{L}})_+ P^c \wedge_{\boxed{(\Sigma^\infty_{\mathbb{L}})_+ GL_1^c(R)}} R$

Model categorically the action of the units on R becomes explicit