Meng Guo 2023-08-14

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What condition we need to have for Y, s.t. $Y \simeq \Omega X$?

Y?? $Y = \Omega X \leftsquigarrow BY$

Thm

Y conn. based. $Y \simeq \Omega X$ for some X iff Y is an A_∞-space.

Def:

An A_∞-structure on a space X consist of an n-tuple of Maps

$$\begin{array}{ccccccccc} X = & E_1 & \subset & E_2 & \subset & \cdots & \subset & E_n \\ & \downarrow p_1 & & \downarrow p_2 & & \cdots & & \downarrow p_n \\ * = & B_1 & \subset & B_2 & \subset & \cdots & \subset & B_n \end{array}$$

$h(-,1) = id$

$h(-,0) = *$

s.t. $\pi_*(E_i, X) \simeq \pi_*(B_i)$, $\forall\ 1 \le i \le n$ $\quad 1 \le n \le \infty$

$X \to E_i \to B_i$ together with a contracting homotopy $h: CE_{n-1} \to E_n$
s.t. $h(CB_{i-1}) \subset B_i$, $\forall i$.

Associahedra ("stasheff operad")

cell complex K_i ($\cong I^{i-2}$)

$\partial K_i = L_i \quad \frac{i(i-1)}{2} - 1$

$X_1 \quad \cdots \quad X_i$ $\qquad$ Cell = Image $(K_r \times K_s)$, $s = i+1-r$

$X_1 \cdots (X_k \cdots X_{k+i-1}) X_{i+k} \cdots X_i$ $\qquad$ $\partial_k(r,s) : K_r \times K_s \to K_{r+s-1}$

Given a way

$$X_1 \cdots (X_k X_{k+1} \cdots X_{k+r-1}) X_{k+r} \cdots X_i \rightsquigarrow \text{cell} \subset L_i = \partial K_i$$

$$\cong$$

$$K_r \times K_s . \quad s = i+1-r$$

Satisfying

$$\begin{tikzcd}
K_r \times K_s \times K_t \arrow[r, "1 \times \partial(s,t)"] \arrow[d, "\partial(r,s) \times 1"'] & K_r \times K_{s+t-1} \arrow[d] \\
K_{r+s-1} \times K_t \arrow[r] & K_{r+s+t-2}
\end{tikzcd}$$

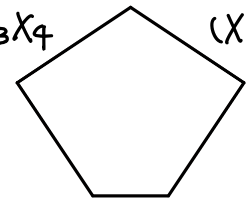
$$\begin{tikzcd}
K_r \times K_s \times K_t \arrow[r, "1 \times T"] \arrow[d, "\partial(r,s)"] & K_r \times K_t \times K_s \arrow[r, "1 \times \partial(t,s)"] & K_r \times K_{t+s-1} \arrow[d] \\
K_{r+s-1} \times K_t \arrow[rr] & & K_{r+t+s-2}
\end{tikzcd}$$

$i = 1$ ✓

$i = 2$, $K_2 = *$

$i = 3$, $K_3 = \dfrac{X_1(X_2X_3)}{(X_1X_2)X_3} = I$

$i=4, \quad X_1X_2X_3X_4, \quad \binom{4}{2}-1=5$



Thm $\quad K_i \cong I^{i-2}$

$K_{n_1} \times K_{n_2}$ one pair

$K_{i_1} \times K_{i_2}$ two pairs

...

An- form

$M_i : K_i \times X^i \to X \quad 2 \le i \le n, \qquad K_2 = *$

(1) $M_2(*, e, x) = M_2(*, x, e)$

(2) $\forall \rho \in K_r, \quad \sigma \in K_s, \quad (r+s=i+1), \quad \rho \times \sigma \subset L_i = \partial K_i$

$$M_i(\underset{\partial_k(r,s)(\rho,\sigma)}{\rho\times\sigma}, x_1, \cdots, x_i) = M_r(\rho, x_1, \cdots, x_{k-1}, M_s(\sigma, x_1, \cdots, x_{r+s-1}), x_{k+s}, \cdots, x_s)$$

(3) $\tau = K_i, \; i > 2$

$$M_i(\tau, x_1, \cdots, x_{j-1}, e, x_{j+1}, \cdots, x_i) = M_{i-1}(S_j(\tau), x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_i)$$

Thm

X with An- structure $\Leftrightarrow (X, M_i)$

Def:

An A_n-space is a space X with A_n-form

or with A_n-structure

$2 \leq n \leq \infty$.

$(X, M_i) \rightsquigarrow \mathcal{E}_i = \coprod_{1 \leq n \leq i-1} K_{n+1} \times X^n / \sim$ $\qquad K_2 \times X^1 \amalg K_3 \times X^2$

forgetting $\downarrow$ $\qquad \downarrow$

first coordinate $B_i = \coprod_{0 \leq n \leq i-2} K_{n+2} \times X^n / \sim$ $\qquad K_2 \times X^0 \amalg K_3 \times X^1$

X - A_∞ $\qquad \mathcal{E}_\infty = \coprod_{n \geq 1} K_{n-1} \times X^n / \sim$

$\downarrow$

"BX" $= B_\infty = \coprod_{n \geq 0} K_{n+2} \times X^n / \sim$

$X \to E_\infty \to B_\infty \rightsquigarrow X \xrightarrow{\simeq} \Omega B_\infty = \Omega BX$

$R = L_{i+1} \times X^i \cup K_{i+1} \times X \times \underbrace{X^{[i-1]}}_{\cap X^{i-1}} \subset K_{i+1} \times X^i$

$\cap X^{i-1}$, one one coordinate $= e$

$(K_{i+1} \times X^i, R) \to (\mathcal{E}_i, \mathcal{E}_{i-1})$

$\alpha|_R$

...

$M_i : K_i \times X^i \to X$

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