Beck's Monadicity Theorem.

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Adjunction: $\mathcal{C}, \mathcal{D}$ be categories.

$$\begin{aligned} F &: \mathcal{C} \to \mathcal{D} \\ G &: \mathcal{D} \to \mathcal{C} \end{aligned}$$ is an adjoint pair, F is left adjoint to G,

denoted by $F \dashv G$, if there are natural isomorphisms

$$\mathcal{D}(Fc, d) \cong \mathcal{C}(c, Gd).$$

for all $c \in \mathcal{C}$, $d \in \mathcal{D}$.

i.e. the functors $\mathcal{D}(F-,-), \mathcal{C}(-,G-) : \mathcal{C}^{op} \times \mathcal{D} \to \mathrm{Set}$

are naturally isomorphic.

Take $d = Fc$, adjunction gives $c \to GFc$.

unit: $\mathrm{Id}_{\mathcal{C}} \Rightarrow GF$.

Similarly, counit $\varepsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$

They satisfy triangle identities.

$$\begin{tikzcd} F \arrow[r, Rightarrow, "F\eta"] \arrow[dr, equal] & FGF \arrow[d, Rightarrow, "\varepsilon F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, Rightarrow, "\eta G"] \arrow[dr, equal] & GFG \arrow[d, Rightarrow, "G\varepsilon"] \\ & G \end{tikzcd}$$

Conversely, we can get $F \dashv G$ if there exists η, ε satisfying the triangle identities.

example: Top_*. based CGWH spaces.

$$\Sigma X = S^1 \wedge X = X \times S^1 / * \times S^1 \vee X \times *$$

$$\Omega X = F(S^1, X) \quad \text{w/ compact open topology.}$$

Then $\Sigma \dashv \Omega$, both in Top_* and $h\mathrm{Top}_*$.

Applying the definition, $\Sigma^n \dashv \Omega^n$.

Def $\mathcal{C}$: category

A <u>monad</u> $T = \langle T, \eta, \mu \rangle$ on $\mathcal{C}$ consists of an endofunctor $T : X \to X$ and natural transformations

$$\eta : I_X \Rightarrow T, \quad \mu : T^2 \Rightarrow T$$

satisfying the following diagrams

$$\begin{tikzcd} T^3 \arrow[r, Rightarrow, "T\mu"] \arrow[d, Rightarrow, "\mu T"'] & T^2 \arrow[d, Rightarrow, "\mu"] \\ T^2 \arrow[r, Rightarrow, "\mu"] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, Rightarrow, "\eta T"] \arrow[dr, equal] & T^2 \arrow[d, Rightarrow, "\mu"] & T \arrow[l, Rightarrow, "T\eta"'] \arrow[dl, equal] \\ & T & \end{tikzcd}$$

<u>Remark</u>: ① Dually, one can define comonad.

② A monad T is a monoid in the category of endofunctors of $\mathcal{C}$.

Every adjunction $F \dashv G$: $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ gives rise to a monad on $\mathcal{C}$ by:

$$T = \langle GF, \eta, G\varepsilon F \rangle$$

($\eta : \mathrm{Id} \Rightarrow GF$, $\varepsilon : FG \Rightarrow \mathrm{Id}$ are unit and counit of the adjunction)

for associativity: diagram chasing.

for unitality:

$$\begin{tikzcd} GF \arrow[r, Rightarrow, "\eta GF"] \arrow[dr, equal] & GFGF \arrow[d, Rightarrow, "G\varepsilon F"] \\ & GF \end{tikzcd}$$

comes from triangle identity

$$\begin{tikzcd} G \arrow[r, Rightarrow, "\eta G"] \arrow[dr, equal] & GFG \arrow[d, Rightarrow, "G\varepsilon"] \\ & G \end{tikzcd}$$

<u>Example</u>. Free monoid functor $F : \mathbf{Set} \to \mathbf{Monoid}$ $\dashv$ forgetful functor $\mathcal{U}$

$T = \mathcal{U}F : \mathbf{Set} \to \mathbf{Set}$ $\quad X \mapsto$ underlying set of free monoid FX

$FX = \{$ strings of lements of X w/ finite length $\}$, w/ concatenation

$\eta_X : X \to TX$, $x \mapsto \langle x \rangle$,

T^2. write string of strings as a string, e.g. $(xy)(zw) \mapsto xyzw$

Question 1: Can every monad be defined from an adjunction?

Answer: Yes.

Def. A **T-algebra** $\langle Y, \theta \rangle$ is a pair consisting of $Y \in \mathcal{C}$ and an arrow $\theta : TY \to Y$ such that the following diagrams commute:

$$\begin{tikzcd} T^2Y \arrow[r, "\mu_Y"] \arrow[d, "T\theta"'] & TY \arrow[d, "\theta"] \\ TY \arrow[r, "\theta"'] & Y \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "\eta_Y"] \arrow[dr, equal] & TY \arrow[d, "\theta"] \\ & Y \end{tikzcd}$$

A **T-algebra morphism** $(A, \alpha) \to (B, \beta)$ is an arrow $f : A \to B$ s.t.

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "\alpha"'] & TB \arrow[d, "\beta"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

The **Eilenberg-Moore category**, denoted by $\mathcal{C}^T$ or $Alg_T(\mathcal{C})$, is the category of T-algebras and morphisms between them

Remark: In view of the morphism, $\theta : TY \to Y$ is a morphism from the free algebra $\langle TY, \mu_Y \rangle$ to $\langle Y, \theta \rangle$.

Theorem 1. If $\langle T, \eta, \mu \rangle$ is a monad on $\mathcal{C}$, then there is an adjunction

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F^T"] & \mathcal{C}^T \arrow[l, shift left, "G^T"] \end{tikzcd}$$

where

$$F^T : \begin{tikzcd} x \arrow[r, mapsto] \arrow[d, "f"'] & \langle Tx, \mu_x \rangle \arrow[d, "Tf"] \\ x' \arrow[r, mapsto] & \langle Tx', \mu_{x'} \rangle \end{tikzcd} \qquad G^T : \begin{tikzcd} \langle Y, \theta \rangle \arrow[r, mapsto] \arrow[d, "f"'] & Y \arrow[d, "f"] \\ \langle Y', \theta' \rangle \arrow[r, mapsto] & Y' \end{tikzcd}$$

with unit $= \eta$ and counit $\varepsilon_{\langle Y, \theta \rangle} : TY \xrightarrow{\theta} Y$

Furthermore, the monad of $F^T \dashv G^T$ is T.

Theorem 1. If $\langle T, \eta, \mu \rangle$ is a monad on $\mathcal{C}$, then there is an adjunction

$$\mathcal{C} \underset{G^T}{\overset{F^T}{\rightleftarrows}} \mathcal{C}^T$$

where

$$F^T: \begin{array}{ccc} x & \longmapsto & \langle Tx, \mu_x \rangle \\ f \downarrow & & \downarrow Tf \\ x' & \longmapsto & \langle Tx', \mu_{x'} \rangle \end{array} \qquad G^T: \begin{array}{ccc} \langle Y, \theta \rangle & \longmapsto & Y \\ f \downarrow & & \downarrow f \\ \langle Y', \theta' \rangle & \longmapsto & Y' \end{array}$$

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Furthermore, the monad of $F^T \dashv G^T$ is T.

Sketch of proof:

① If $x \in \mathcal{C}$, $\langle Tx, \mu_x \rangle$ is a T-algebra.

② Define the unit and counit as claimed, check the triangle identities.

e.g.

$$\begin{tikzcd} G^T \arrow[r, Rightarrow, "\eta G^T"] \arrow[dr, equal] & G^T F^T G^T \arrow[d, Rightarrow, "G^T \varepsilon"] \\ & G^T \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} Y \arrow[r, "\eta_Y"] \arrow[dr, equal] & TY \arrow[d, "\theta"] \\ & Y \end{tikzcd}$$

③. Check $\langle G^T F^T, \eta, G^T \varepsilon F^T \rangle = \langle T, \eta, \mu \rangle$. $\square$

Question 2: Start with $F \dashv G$: $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$, construct $T = GF$. Then how is $\mathcal{C}^T$ related to $\mathcal{D}$?

In $\mathbf{Set} \rightleftarrows \mathbf{Monoid}$, $\mathbf{Set}^T$ looks exactly like $\mathbf{Monoid}$.

Theorem 2. In the context above, there exists a functor $G' : \mathcal{D} \to \mathcal{C}^T$ such that $G = G^T G'$.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, shift left, "F^T"] \arrow[rr, bend left=40, "F"] & \mathcal{C}^T \arrow[l, shift left, "G^T"] \arrow[r, shift left, dashed, "F'"] & \mathcal{D} \arrow[l, shift left, "G'"] \arrow[ll, bend left=40, "G"]
\end{tikzcd}$$

Sketch of proof: For $d \in \mathcal{D}$,

$\langle Gd, G\varepsilon \rangle$ is a T-algebra.

Define $G'd = \langle Gd, G\varepsilon \rangle$, $G'(f) = Gf$. $\square$

Beck's Theorem tells us when $\mathcal{C}^T$ will be isomorphic/equivalent to $\mathcal{D}$.

Def. We say $F \dashv G$ is <u>monadic</u> if $G' : \mathcal{D} \to \mathcal{C}^T$ is an equivalence of categories. We say a functor $G : \mathcal{D} \to \mathcal{C}$ is <u>monadic</u> if there exists a monadic adjunction $F \dashv G$.

Motivation. In $\mathrm{Set} \underset{U}{\overset{F}{\rightleftarrows}} \mathrm{Monoid}$. If Y is a T-algebra (as a set), how to construct a monoid out of Y?

Answer: construct quotient of the free monoid TY.

Theorem 3. If $\mathcal{D}$ has coequalizers, then there exists a functor $F' : \mathcal{C}^T \longrightarrow \mathcal{D}$ such that $F' \dashv G'$. (the hypothesis can be weakened)

Theorem 4 (Beck). An adjunction $F \dashv G$ is monadic if and only if $G : \mathcal{D} \to \mathcal{C}$ creates coequalizers of <u>G-split pairs</u>.

Hence let's look at some special coequalizers.

<u>Def</u>. In a category $\mathcal{C}$, a <u>split coequalizer</u> (fork) is given by the diagram

$$\begin{tikzcd} X \arrow[r, shift left=2, "f"] \arrow[r, "g" description] & Y \arrow[l, shift left=2, "j"] \arrow[r, shift left, "q"] & Z \arrow[l, shift left, "i"] \end{tikzcd}$$

such that

$$qf = qg, \quad qi = id_Z, \quad fj = iq, \quad gj = id_Y.$$

<u>Prop</u>. Split coequalizers are coequalizers.

<u>Proof</u>:

$$\begin{tikzcd} X \arrow[r, shift left=2, "f"] \arrow[r, "g" description] & Y \arrow[l, shift left=2, "j"] \arrow[r, shift left, "q"] \arrow[dr, "h"'] & Z \arrow[l, shift left, "i"] \arrow[d, dashed, "h\circ i"] \\ & & Z' \end{tikzcd}$$

$hf = hg$

$\Rightarrow hfj = hgj$

$\Rightarrow hiq = h$

If $kq = h$, $k = kqi = hi$ $\square$

The image of any split coequalizer is still a split coequalizer.

This structure is encoded in the data of a $T = \langle FG, \eta, G\varepsilon F\rangle$-algebra.

$$\begin{tikzcd} TTY \arrow[r, shift left=2, "\mu_Y"] \arrow[r, "T\theta" description] & TY \arrow[l, shift left=2, "\eta_{TY}"] \arrow[r, shift left, "\theta"] & Y \arrow[l, shift left, "\eta_Y"] \end{tikzcd}$$

e.g. η is a natural transformation $Id \Rightarrow T$. thus.

$$\begin{tikzcd} TY \arrow[r, "\eta_{TY}"] \arrow[d, "\theta"'] & T^2Y \arrow[d, "T\theta"] \\ Y \arrow[r, "\eta_Y"'] & TY \end{tikzcd}$$

Furthermore, this coequalizer tells us how to construct Y from TY as a quotient/coequalizer:

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F^T"] \arrow[rr, bend left=50, "F"] & \mathcal{C}^T \arrow[l, shift left, "G^T"] \arrow[r, shift left, dashed, leftarrow, "F'"] & \mathcal{D} \arrow[l, shift left, "G'"] \arrow[ll, bend left=50, "G"] \end{tikzcd}$$

If $Y \in \mathcal{C}^T$, define $F'Y$ as the coequalizer of

$$FTG^TY \underset{FG^T\theta}{\overset{\varepsilon_{FG^TY}}{\rightrightarrows}} FG^TY$$

algebra map of FG^TY is μ.

This is the construction in Theorem 3.

Def. A <u>G-split pair</u> is a pair of maps $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ in $\mathcal{D}$ together w/ a split coequalizer $GX \underset{Gg}{\overset{Gf}{\rightrightarrows}} GY \underset{i}{\rightleftarrows} C$ in $\mathcal{C}$.

We say $G: \mathcal{D} \to \mathcal{C}$ <u>creates coequalizers of G-split pairs</u> if

(1) for every G-split pair $X \underset{g}{\overset{f}{\rightrightarrows}} Y$, there is a fork in $\mathcal{D}$

$$X \rightrightarrows Y \longrightarrow D$$

such that $Gq \circ i$ is an isomorphism. $\Big(\underbrace{Gq \circ i \text{ is the universal arrow } C \to GD}\Big)$

(2). any diagram satisfying (1) is a coequalizer.

We say $G: \mathcal{D} \to \mathcal{C}$ <u>strictly</u> creates coequalizers of G-split pairs if the lift in $\mathcal{D}$ is unique.

Prop. For any monad T on a category $\mathcal{C}$, the forgetful functor $G^T : \mathcal{C}^T \to \mathcal{C}$ strictly creates coequalizers of G^T-split pairs.

<u>sketch of proof</u>: Start with $A \rightrightarrows B$ in $\mathcal{C}^T$ together with

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, shift left, "r"] \arrow[l, bend left] & C \arrow[l, shift left] \end{tikzcd} \text{ in } \mathcal{C}.$$

want to lift C to an object of $\mathcal{C}^T$.

Look at $TA \rightrightarrows TB \to TC$

Define $TC \to C$ as the universal arrow for

$$\begin{tikzcd} TA \arrow[r, shift left] \arrow[r, shift right] & TB \arrow[r] \arrow[d, "\beta"'] & C \\ & B \arrow[ur, "r"'] & \end{tikzcd}$$

since we want r be algebra map.

Check that it's a coequaliser diagram.

uniqueness due to G^T is a forgetful functor. $\square$

<u>Cor.</u> If $F \dashv G$ monadic, then G creates coequalizers of G-split pairs.

<u>Proof</u>: $G = G^T G'$. $\square$

<u>Sketch proof of Theorem 4</u>:

If G creates coequalizers of G-split pairs.

If Y is a T-algebra, $G^T Y \in \mathcal{C}$

$FTY^u \rightrightarrows FY^u$ is in $\mathcal{D}$. $F'Y$ is coequalizer.

with split fork $T^2 Y^u \rightrightarrows TY^u \longrightarrow Y^u$ in $\mathcal{C}$.

Then $Y^u \to G'F'Y^u$ is an isomorphism.

remains to prove $F'G' \Rightarrow \mathrm{Id}$ is an iso. left as an exercise

$\square$.