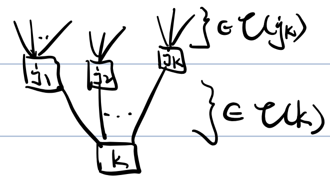
1. Basic Definitions

Lets get into the formal def's.

Def Our base category will be pointed spaces.

An operad $\mathcal{C}$ consists of unpted spc. $\mathcal{C}(j)$ w/ right action by Σ_j for $j \geq 0$, $\mathcal{C}(0) = \{*\}$, a choice of $1 \in \mathcal{C}(1)$ & cts fcns (composition laws)

$$\gamma : \mathcal{C}(k) \otimes \mathcal{C}(j_1) \otimes \cdots \otimes \mathcal{C}(j_k) \longrightarrow \mathcal{C}(j_1 + \cdots + j_k)$$

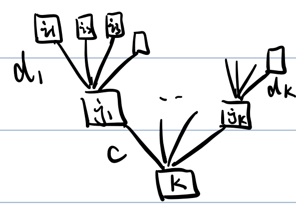


with the following data:

① Associativity: $\forall c \in \mathcal{C}(k)$, $d_s \in \mathcal{C}(j_s)$, $e_t \in \mathcal{C}(i_t)$

$$\gamma(\gamma(c, d_1, \cdots d_k); e_1, \cdots e_j) = \gamma(c, f_1, \cdots f_k)$$

where $f_s = \gamma(d_s; e_{j_1 + \cdots + j_{s-1} + 1}, \cdots e_{j_1 + \cdots + j_s})$



② Identity: $\gamma(1; d) = d \quad \forall d \in \mathcal{C}(j)$

$\gamma(c; 1, 1, \cdots 1) = c \quad \forall c \in \mathcal{C}(k)$

③ Σ_j-action equivariantly:

$\forall c \in \mathcal{C}(k)$, $d_s \in \mathcal{C}(j_s)$, $\sigma \in \Sigma_k$, $\tau_s \in \Sigma_{j_s}$

$$\gamma(c\sigma; d_1, \cdots d_k) = \gamma(c; d_{\sigma^{-1}(1)}, \cdots d_{\sigma^{-1}(k)})\sigma(j_1, \cdots j_k)$$

permutation of j letters which perm. the k-blocks of letters

& $\gamma(c, d_1\tau_1, \cdots d_k\tau_k) = \gamma(c; d_1, \cdots d_k)\underbrace{(\tau_1 \oplus \cdots \oplus \tau_k)}_{\text{evident inclusion } \Sigma_j \longrightarrow \Sigma_{j_1} \times \cdots \times \Sigma_{j_k}}$

Def A morphism of operads is a seq. of Σ_j-equivariant maps $\psi_j : \mathcal{C}(j) \to \mathcal{C}'(j)$ s.t. $\psi_1(1) = 1'$ &

$$\begin{array}{ccc} \mathcal{C}(k) \otimes \mathcal{C}(j_1) \otimes \cdots \otimes \mathcal{C}(j_k) & \xrightarrow{\gamma} & \mathcal{C}(j) \\ \downarrow {\scriptstyle \psi_k \times \psi_{j_1} \times \cdots \times \psi_{j_k}} & & \downarrow {\scriptstyle \psi_j} \\ \mathcal{C}'(k) \otimes \mathcal{C}'(j_1) \otimes \cdots \otimes \mathcal{C}'(j_k) & \xrightarrow{\gamma'} & \mathcal{C}'(j) \end{array}$$

E.g.: ① Prototypical example: endomorphism operad in pted spaces.

$\forall X \in Top_*$

define $\mathcal{E}_X(j) := \mathrm{Hom}_{Top_*}(X^{\wedge j} \longrightarrow X)$ $\qquad \mathcal{E}_X(0) = \{* \longrightarrow X\}$.

unit $\in \mathrm{Hom}(X, X)$ is the identity map

the right action of Σ_j is given by permuting $X^{\otimes j}$.

② Comm. operad: $\mathcal{N}(k) = *$ for all $k \geq 0$.

Σ_j-actions are trivial, γ are evident identifications.

③ Assoc. operad: $\mathcal{M}(k) := \Sigma_k$ $\quad k \geq 1$

the composition

$$\Sigma_k \times \Sigma_{j_1} \times \cdots \times \Sigma_{j_k} \longrightarrow \Sigma_j$$

$(\sigma, \tau_1, \cdots \tau_k) \longmapsto$ permute blockwisely

Def Given an operad $\mathcal{C}$, an algebra X over $\mathcal{C}$ in Top_* is a morphism

$$\theta : \mathcal{C} \longrightarrow \mathcal{E}_X$$

equivalently, $X \in Top_*$ together w/ Σ_j-equiv. maps $\mathcal{C}(j) \times X^{\wedge j} \xrightarrow{\theta_j} X$

that are cptible w/ operadic stru. i.e.

①
$$\begin{tikzcd}[column sep=small]
\mathcal{C}(k)\times\mathcal{C}(j_1)\times\cdots\times\mathcal{C}(j_k)\times X^{j} \arrow[r, "\gamma\times 1"] \arrow[d, "\text{permute}"] & \mathcal{C}(j)\times X^{j} \arrow[dr, "\theta_j"] & \\
\mathcal{C}(k)\times\mathcal{C}(j_1)\times X^{j_1}\times\cdots\times\mathcal{C}(j_k)\times X^{j_k} \arrow[r, "1\times\theta_{j_1}\times\cdots\times\theta_{j_k}"] & \mathcal{C}(k)\times X^{k} \arrow[r, "\theta_k"] & X
\end{tikzcd}$$

② $\theta_1(1;x)=x$ for $x\in X$

③ $\theta_j(c\sigma;y)=\theta_j(c;\sigma y)$ $\forall c\in\mathcal{C}(j)$ $\sigma\in\Sigma_j$. $y\in X^j$

An alg. struct. give a way of parametrizing n-ary operations on X.

Eg: ① Apparently $X\in\mathcal{C}_X$-alg.

② $\mathcal{M}$-alg are associative monoids.

③ $\mathcal{N}$-alg are comm. monoids.

$\boxed{\text{2. Comparison w/ monad}}$

Recall the definition of a monad (C,η,μ) in a cat. $\mathcal{S}$ is a functor $C:\mathcal{S}\longrightarrow\mathcal{S}$ + natural trans:

$$\mu: C^2\longrightarrow C \quad \& \quad \eta: \mathrm{id}\longrightarrow C$$

s.t.

$$\begin{tikzcd}
CX \arrow[r, "C(\eta X)"] \arrow[dr, equal] & C^2X \arrow[d, "\mu"] & CX \arrow[l, "\eta(CX)"'] \arrow[dl, equal] \\
& CX &
\end{tikzcd}
\quad \& \quad
\begin{tikzcd}
C^3X \arrow[r, "\mu(CX)"] \arrow[d, "C(\mu X)"'] & C^2X \arrow[d, "\mu_X"] \\
C^2X \arrow[r, "\mu_X"] & CX
\end{tikzcd}$$

commute for $\forall X$.

<u>Def</u>| An alg. (X,ξ) over a monad (C,η,μ) is $\xi: CX\longrightarrow X$ s.t.

$$\begin{tikzcd}
X \arrow[r, "\eta"] \arrow[dr, equal] & CX \arrow[d, "\xi"] \\
& X
\end{tikzcd}
\quad \& \quad
\begin{tikzcd}
C^2X \arrow[r, "\mu"] \arrow[d, "C\xi"'] & CX \arrow[d, "\xi"] \\
CX \arrow[r, "\xi"] & X
\end{tikzcd}$$

Construction

Given an operad $\mathcal{C}$, construct a monad C assoc. to $\mathcal{C}$ by

$$CX := \coprod_{j\geq 0} \mathcal{C}(j) \times_{\Sigma_j} X^j / \sim$$

where the relations $\sim$ are generated by

$\forall\, c \in \mathcal{C}(j)\ \ y \in X^{j-1}$, $\quad (\sigma_i c, y) \sim (c, s_i y)$

$$\sigma_i : \mathcal{C}(j) \longrightarrow \mathcal{C}(j-1) \quad \text{by} \quad \sigma_i c = \gamma(c; 1^i \times * \times 1^{j-1-i}) \in \mathcal{C}(1)^i \times \mathcal{C}(0) \times \mathcal{C}(1)^{j-1-i}$$

$$s_i : X^{j-1} \longrightarrow X^j \quad \text{by} \quad (x_1, \cdots x_{j-1}) \longmapsto (x_1, \cdots x_i, *, x_{i+1}, \cdots x_{j-1})$$

The unit $\eta : X \to CX$ is $\eta \otimes \mathrm{id} : * \otimes X \longrightarrow \mathcal{C}(1) \otimes X$

$\mu : CCX \longrightarrow CX$ is induced by the following maps

$$\begin{array}{c} \mathcal{C}(k) \otimes \mathcal{C}(j_1) \otimes X^{j_1} \otimes \cdots \otimes \mathcal{C}(j_k) \otimes X^{\otimes j_k} \\ \downarrow \text{shuffle} \\ \mathcal{C}(k) \otimes \mathcal{C}(j_1) \otimes \cdots \otimes \mathcal{C}(j_k) \otimes X^{\otimes j} \\ \downarrow \gamma \otimes \mathrm{id} \\ \mathcal{C}(j) \otimes X^j \end{array}$$

The topology on CX can be made clear as follows:

Consider $F_k CX :=$ image of $\coprod_{j=0}^{k} \mathcal{C}(j) \times_{\Sigma_j} X^j$

Fact: $F_{k-1} CX$ is a closed subspace of $F_k CX$, in particular, a cofibration.

$F_0 CX$ can be taken as basept of CX.

$CX =$ union of topology $\bigcup_{j\geq 0} F_j CX$.

It's a $\mathcal{C}$-alg. by def.

This CX is called the free $\mathcal{C}$-alg. gen. by X, i.e.

We have an adjunction

$$C : Top_* \rightleftarrows \mathcal{C}\text{-alg.} : U \quad (\perp)$$

$$\mathrm{Hom}_{\mathcal{C}\text{-alg.}}(CX, Y) \cong \mathrm{Hom}_{Top_*}(X, UY)$$

Prop Let $\mathcal{C}$ be an operad & C be its assoc. monad. $\exists$ a 1-1 correspondence between $\mathcal{C}$-actions $\{\theta_j : \mathcal{C}(j) \otimes X^{\otimes j} \longrightarrow X\}_{j \geq 0}$ & C-alg on X $\xi : CX \longrightarrow X$ iff the following diagrams commutes:

$$\begin{tikzcd} \mathcal{C}(j) \otimes X^{\otimes j} \arrow[rr, "\pi_j"] \arrow[dr, "\theta_j"'] & & CX \arrow[dl, "\xi"] \\ & X & \end{tikzcd}$$

proof: maps $\theta_j : \mathcal{C}(j) \otimes_{\Sigma_j} X^{\otimes j} \longrightarrow X$ together specifies a

$\xi : CX \longrightarrow X$

E.g.: in Top_*, MX is the James construction $\Omega\Sigma X$

NX is the infinite symmetric product on X

$\pi_*(NX) \cong \tilde{H}_*(X)$