James construction and the Hilton-Milnor Theorem

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Aug 14, 2023

Recall that the associated monad C of an operad $\mathscr{C}$ is defined by the following.

Definition 1. $CX = \coprod \mathscr{C}(n) \times_{\Sigma_n} X^n / \sim$ where $(\sigma_i c, x) \sim (c, s_i x)$.

Consider the filtration $F_r CX$ which is the image of $\coprod_{j=0}^{r} \mathscr{C}(j) \times X^j$. We have

$$F_r CX / F_{r-1} CX \mathscr{C}(r)_+ \wedge_{\Sigma_r} X^{\wedge r}.$$

Theorem 2 (Recognition principle)**.** *There exists Σ-free operad $\mathscr{C}_n$ such that every n-fold loop space is a $\mathscr{C}_n$-space, and every connected $\mathscr{C}_n$-space has the weak homotopy type of an n-fold loop space.*

Theorem 3 (approximation theorem)**.** *Let C_n be the monad associated to the $\mathscr{C}_n$ in the approximation theorem. Then*

$$\alpha_n : C_n \to \Omega^n \Sigma^n$$

is a weak equivalence if X is connected.

Let M be the associative monad. Then

$$MX = \coprod X^r / \sim$$

where

$$(x_1, \dots, x_n) \sim (x_1, \dots, x_i, *, x_{i+1}, \dots, x_n).$$

MX is called the James construction or James reduced product.

When $n = 1$, we have

$$MX \xleftarrow{\epsilon} C_1 X \xrightarrow{\alpha} \Omega \Sigma X$$

where the map ϵ is a homotopy equivalence and α is a weak equivalence if X is connected.

Theorem 4 (James splitting)**.** *Assume X is connected. Then (weak equivalence)*

$$\Sigma \Omega \Sigma X \simeq \bigvee_{i \geq 1} \Sigma X^{\wedge i}.$$

Proof. It suffices to prove that $\Sigma MX \simeq \bigvee_{i \geq 1} \Sigma X^{\wedge i}$.

There are $\binom{r}{q}$ maps $X^r \to X^q \to X^{\wedge q}$ which defines

$$j_{qr} : X^r \to (X^{\wedge q})^m$$

where $m = \binom{r}{q}$. This map is trivial if $r < q$ and $\binom{r}{q} = 0$.

The following commutative diagram

$$\begin{array}{ccc} X^r & \xrightarrow{\phi} & X^{r+1} \\ \downarrow & & \downarrow \\ (X^{\wedge q})^m & \xrightarrow{\bar{\phi}} & (X^{\wedge q})^n \end{array}$$

$(n = \binom{r+1}{q})$ means that we can use j_{qr} for various r to define map $j_q : MX \to M(X^{\wedge q})$.

Next we consider define the map k_r by the composition

$$MX \xrightarrow{\Delta} (MX)^r \to (MX)^r \xrightarrow{\prod j_q} \prod_{q=1}^{r} M(X^{\wedge q}) \to \prod_{q=1}^{r} M(\bigvee_{p=1}^{r} X^{\wedge p}) \xrightarrow{\mu} M(\bigvee_{q=1}^{r} X^{\wedge q})$$

The last map comes from the multiplication in MY because MY is a monoid for any Y.

The left square of the following commutative diagram

$$\begin{array}{ccccc} F_{r-1}MX & \longrightarrow & F_rMX & \longrightarrow & X^{\wedge r} \\ \downarrow{\scriptstyle k_{r-1}} & & \downarrow{\scriptstyle k_r} & & \downarrow{\scriptstyle \eta} \\ M(\bigvee_{q=1}^{r-1} X^{\wedge q}) & \longrightarrow & M(\bigvee_{q=1}^{r} X^{\wedge q}) & \longrightarrow & MX^{\wedge r} \end{array}$$

implies that we can define a map $k_\infty : MX \to M(\bigvee_{q \geq 1} X^{\wedge q})$.

Consider the adjoint map $\tilde{k}_r : \Sigma F_rMX \to \Sigma(\bigvee_{q=1}^{r} X^{\wedge q})$ of the following composition

$$F_rMX \to MX \xrightarrow{k_r} M(\bigvee_{q=1}^{r} X^{\wedge q}) \xrightarrow{\eta} \Omega\Sigma(\bigvee_{q=1}^{r} X^{\wedge q}).$$

We claim that $\tilde{k}_r$ is a homotopy equivalence. It can be proved by induction by the following commutative diagram

$$\begin{array}{ccccccc} X^{\wedge r} & \longrightarrow & \Sigma F_{r-1}MX & \longrightarrow & \Sigma F_rMX & \longrightarrow & \Sigma X^{\wedge r} \\ & & \downarrow{\scriptstyle \tilde{k}_{r-1}} & & \downarrow{\scriptstyle \tilde{k}_r} & & \| \\ & & \Sigma(\bigvee_{q=1}^{r-1} X^{\wedge q}) & \longrightarrow & \Sigma(\bigvee_{q=1}^{r} X^{\wedge q}) & \longrightarrow & \Sigma X^{\wedge r} \end{array}$$

By taking the limit $\tilde{k}_\infty : \Sigma MX \to \Sigma(\bigvee_{q \geq 1} X^{\wedge q})$ is also a homotopy equivalence. □

Remark 5. The theorem and proof can be generalized to Λ-spaces $\mathbf{X}$ which are sequences of spaces $\mathbf{X}_r$ with some structures that we used in the proof and X^r can be replaced with $\mathbf{X_r}$.

Theorem 6 (Hilton-Milnor Theorem)**.** *Assume that X, Y are based space. We have*

$$\Omega(\Sigma X \vee \Sigma Y) \simeq \Omega\Sigma X \times \Omega\Sigma Y \Omega\Sigma(\bigvee_{i,j \geq 1} X^{\wedge i} \wedge Y^{\wedge j})$$

Consider the homotopy fiber F of $X \vee Y \to Y$:

$$F \to X \vee Y \to Y$$

Because the collapsing map $X \vee Y \to Y$ has a section we have

$$\Omega X \vee Y \simeq \Omega Y \times \Omega F$$

Exercise 7. Prove that

$$F \simeq X \times \Omega Y / \{*\} \times \Omega Y \simeq X \vee (X \wedge \Omega Y)$$

Hint: Use the definition of homotopy fiber and construct maps between both sides.

Exercise 8. Use the James splitting and the previous exercise to prove the Hilton-Milnor Theorem.