1 Equivariant Thom-Pontryagin construction

Goal: Explain the following diagram of equivariant homology theories:

\[
\begin{array}{ccc}
\mathcal{N}^G_* & \xrightarrow{\Theta^G} & \text{mO}^G_* \\
\downarrow & & \downarrow \\
\mathcal{N}^{G:S}_* & \xrightarrow{\Theta^G} & \text{MO}^G_*
\end{array}
\Rightarrow
\begin{array}{ccc}
\mathcal{N}^G_* & \xrightarrow{\Theta^G} & \text{mOP}^G_* \\
\downarrow & & \downarrow \\
\mathcal{N}^{G:S}_* & \xrightarrow{\Theta^G} & \text{MOP}^G_*
\end{array}
\]

In this diagram:

1. \text{MO} is the ultra-commutative Thom spectrum, \text{mO} is an \(E_8\)-Thom spectrum. \text{MOP} and \text{mOP} are periodic extensions of \text{MO} and \text{mO}, respectively.

2. The vertical transformation in the middle column is an isomorphism for \(G = e\). This is not true in general.

3. \(\mathcal{N}^G_*\) is a geometrically defined equivariant bordism and \(\mathcal{N}^{G:S}_*\) is a stable equivariant bordism. They are not represented by orthogonal spectra, but defined from bordism classes of \(G\)-manifolds.

4. The two \(\Theta^G\) maps are equivariant Thom-Pontryagin construction and its stabilization". The upper \(\Theta^G\) is an isomorphism when \(G\) is a product of finite groups and a torus.

1.1 Global Thom spectra

We first define the global Thom spectra: \(\text{MGr}, \text{MOP}, \text{MO}, \text{mOP}\) and \(\text{mO}\).

**Example 1.1.** We start with \(\text{MGr}\), the Thom spectrum over the additive Grassmannian \(\text{Gr}\). The value of \(\text{Gr}\) at each inner product space \(V\) is

\[
\text{Gr}(V) = \bigsqcup_{n \geq 0} \text{Gr}_n(V).
\]

The total space of the tautological Euclidean vector bundle (of no constant rank) over \(\text{Gr}(V)\) consists of points \((x, U)\) such that \(x \in U \in \text{Gr}(V)\). We define \(\text{MGr}(V)\)
to be the Thom space of this tautological bundle over $\text{Gr}(V)$. The structure maps are given by

$$O(V, W) \wedge \text{MGr}(V) \to \text{MGr}(W)$$

$$(w, \varphi) \wedge (x, U) \mapsto (w + \varphi(x), \varphi^\perp \oplus \varphi(U)),$$

where $\varphi^\perp$ is the orthogonal complement of the image of $\varphi : V \to W$. Multiplication maps are defined by direct sum:

$$\mu_{V, W} : \text{MGr}(V) \wedge \text{MGr}(W) \to \text{MGr}(V \oplus W)$$

$$(x, U) \wedge (x', U') \mapsto ((x, x'), (U, U')).$$

Unit maps are defined by

$$\eta(V) : S^V \to \text{MGr}(V), \quad v \mapsto (v, V).$$

The multiplication maps are binatural, associative, commutative and unital, making $\text{MGr}$ an ultra-commutative ring spectrum. $\text{MGr}$ is graded, with the $k$-th homogeneous summand given by

$$\text{MGr}^{[k]}(V) = \text{Th} \left( \text{Gr}_{|V|+k}(V) \right).$$

This shows $\text{MGr}$ is concentrated in non-positive degrees and the unit morphism $\eta : S \to \text{MGr}$ is an isomorphism onto $\text{MGr}^{[0]}$. Let $V$ be a representation of a compact Lie group $G$, we define the inverse Thom class $\tau_{G, V} \in \text{MGr}_G^G(S^V)$ as a class represented by the $G$-map:

$$t_{G, V} : S^V \to \text{Th}(\text{Gr}(V)) \wedge S^V = \text{MGr}(V) \wedge S^V$$

$$v \mapsto (0, \{0\}) \wedge (-v).$$

The internal degree of $\tau_{G, V}$ is equal to $-\dim V$.

**Example 1.2.** We define two ultra-commutative ring spectra $\text{MO}$ and $\text{MOP}$. $\text{MOP}$ is a Thom spectra over the orthogonal space $BOP$, whose value at an inner product space $V$ is

$$BOP(V) = \bigcap_{n \geq 0} \text{Gr}_n(V^2).$$

Define $\text{MOP}(V)$ as the Thom space of the tautological vector bundle over $BOP(V)$. The structure maps are given by

$$O(V, W) \wedge \text{MOP}(V) \to \text{MOP}(W)$$

$$(w, \varphi) \wedge (x, U) \mapsto ((w, 0) + \text{BOP}(\varphi)(x), \text{BOP}(\varphi)(U)).$$

Multiplication maps are defined by

$$\mu_{V, W} : \text{MOP}(V) \wedge \text{MOP}(W) \to \text{MOP}(V \oplus W)$$

$$(x, U) \wedge (x', U') \mapsto (\kappa_{V, W}(x, x'), \kappa_{V, W}(U \oplus U')).$$
where $\kappa_{V,W} V^2 \oplus W^2 \sim (V \oplus W)^2$ is the preferred isometry defined by

$$
\kappa_{V,W}((v, v'), (w, w')) = ((v, w), (v', w')).
$$

Unit maps are defined by

$$
\eta^V : S^V \to \text{MOP}(V), \quad v \mapsto ((v, 0), (V, 0)).
$$

The multiplication maps make $\text{MOP}$ an ultra-commutative ring spectrum. The orthogonal space $\text{BOP}$ is $\mathbb{Z}$-graded, with $k$-th homogeneous summand

$$
\text{BOP}^k(V) = Gr_{|V|+k}(V^2).
$$

The Thom spectra $\text{MOP}$ inherits the $\mathbb{Z}$-grading from $\text{BOP}$. $\text{MOP}(V)$ is the wedge sum of the Thom spaces $\text{MOP}^k(V)$ for $-|V| \leq k \leq |V|$ and thus

$$
\text{MOP} = \bigvee_{k \in \mathbb{Z}} \text{MOP}^k.
$$

We define $\text{MO} = \text{MOP}^{[0]}$. It is an ultra-commutative ring spectrum on its own right. Explicitly, $\text{MO}(V)$ is the Thom space of the tautological vector bundle over $Gr_{|V|}(V^2)$.

**Example 1.3.** We define two $E_8$-ring spectra $\text{mO}$ and $\text{mOP}$, the Thom spectra over the orthogonal spaces $\text{bO}$ and $\text{bOP}$. The value of $\text{bOP}$ at an inner product space $V$ is

$$
\text{bOP}(V) = \bigoplus_{n \geq 0} Gr_n(V \oplus \mathbb{R}^\infty),
$$

For a linear isometric embedding $\varphi : V \to W$, the induced map $\text{bOP}(\varphi) : \text{bOP}(V) \to \text{bOP}(W)$ is defined as

$$
\text{bOP}(\varphi)(L) = (\varphi \oplus \mathbb{R}^\infty)(L) + ((W - \varphi(V)) \oplus 0).
$$

Over $\text{bOP}(V)$ sits a tautological Euclidean vector bundle (of non-constant rank) and we define $\text{mOP}(V)$ as the Thom space of this tautological bundle. The structure maps are given by

$$
O(V, W) \wedge \text{mOP}(V) \longrightarrow \text{mOP}(W)
$$

$$
(w, \varphi) \wedge (x, U) \longrightarrow ((w, 0) + \text{bOP}(\varphi)(x), \text{bOP}(\varphi)(U)).
$$

The $E_8$-structures on $\text{mO}$ and $\text{mOP}$ are inherited from those on $\text{bO}$ and $\text{bOP}$ by the linear isometry operad. Multiplication maps are defined by

$$
\mu_{V,W} : L \wedge \text{mOP}(V) \wedge \text{mOP}(W) \longrightarrow \text{mOP}(V \oplus W)
$$

$$
\psi \wedge (x, U) \wedge (x', U') \longrightarrow (\psi_\uparrow(x, x'), \psi_\uparrow(U \oplus U')),
$$

where $\psi_\uparrow$ is the linear isometric embedding

$$
\psi_\uparrow : V \oplus \mathbb{R}^\infty \oplus W \oplus \mathbb{R}^\infty \longrightarrow V \oplus W \oplus \mathbb{R}^\infty
$$

$$
(v, y, w, z) \longrightarrow (v, w, \psi(y, z)).
$$
Unit maps are defined by
\[ \eta^V : S^V \to \mathbf{mOP}(V), \quad v \mapsto ((v, 0), (V \oplus 0)). \]

\( \mathbf{mOP} \) is \( \mathbb{Z} \)-graded, \( \mathbf{mOP}^{[k]}(V) \) is the Thom space of the tautological bundle over \( \mathbf{bOP}^{[k]}(V) = Gr_{V+V}'(V \oplus \mathbb{R}^\infty) \). Then \( \mathbf{mOP}(V) \) is the wedge sum of \( \mathbf{mOP}^{[k]}(V) \) for \( |V| + k \geq 0 \) and there is a decomposition
\[ \mathbf{mOP} = \bigvee_{k \in \mathbb{Z}} \mathbf{mOP}^{[k]}. \]

We define \( \mathbf{mO} = \mathbf{mOP}^{[0]} \) to the zeroth summand in this decomposition.

The equivariant cohomology theories represented by the global Thom spectra are related by the following:
\[ \mathbf{MGr}_G^G(A) \xrightarrow{\text{invert } \tau_{G,V}} \mathbf{mOP}_G^G(A) \xrightarrow{\text{invert all } \tau_{G,V}} \mathbf{MOP}_G^G(A) \]

More precisely, we define maps \( a : \mathbf{MGr} \to \mathbf{mOP} \) and \( b : \mathbf{MGr} \to \mathbf{mOP} \), whose values at an inner product space \( V \) are
\[ a(V) : \mathbf{MGr}(V) \to \mathbf{mOP}(V) \quad (x, L) \mapsto ((x, 0), (V \oplus 0)), \]
\[ b(V) : \mathbf{MGr}(V) \to \mathbf{mOP}(V) \quad (x, L) \mapsto ((x, 0), (L \oplus 0)). \]

The localized \( \mathbf{MOP} \) and \( \mathbf{mOP} \) are defined by
\[ \mathbf{MGr}^G_k(A)[1/\tau_{G,V}] = \colim_{V \in \mathcal{V}(U_k)} \mathbf{MGr}^G_k(A \wedge S^V) \]
\[ \mathbf{MGr}^G_k(A)[1/\tau_{G,R}] = \colim_{n \geq 0} \mathbf{MGr}^G_k(A \wedge S^n), \]
where the structure maps are
\[ \mathbf{MGr}^G_k(A \wedge S^V) \xrightarrow{\tau_{G,V}} \mathbf{MGr}^G_k(A \wedge S^V \wedge S^{W-V}) \cong \mathbf{MGr}^G_k(A \wedge S^W), \]
\[ \mathbf{MGr}^G_k(A \wedge S^n) \xrightarrow{\tau_{G,R}} \mathbf{MGr}^G_k(A \wedge S^n \wedge S^R) \cong \mathbf{MGr}^G_k(A \wedge S^{n+1}). \]

**Theorem 1.4.** The maps \( a \) and \( b \) are compatible with the colimits and they assemble into maps
\[ a^\sharp : \mathbf{MGr}^G_k(A)[1/\tau_{G,V}] \to \mathbf{MOP}^G_k(A), \]
\[ b^\sharp : \mathbf{MGr}^G_k(A)[1/\tau_{G,R}] \to \mathbf{mOP}^G_k(A). \]

The maps \( a^\sharp \) and \( b^\sharp \) are isomorphisms for every compact Lie group \( G \), based \( G \)-space \( A \) and integer \( k \).
1.2 Geometric equivariant bordism

Definition 1.5. Let $G$ be a compact Lie group and $X$ be a $G$-space. A singular $G$-manifold over $X$ is a pair $(M, h)$, where $M$ is a closed smooth $G$-manifold and $h : M \to X$ is a continuous $G$-map. Two singular $G$-manifolds $(M, h)$ and $(M', h')$ are bordant if there is a triple $(B, H, \psi)$, where $B$ is a compact smooth $G$-manifold, $H : B \to X$ is continuous $G$-map, and $\psi$ is an equivariant diffeomorphism:

$$\psi : M \cup M' \to \partial B$$

such that $(H \circ \psi)|_M = h$ and $(H \circ \psi)|_{M'} = h'$.

Bordism of singular $G$-manifolds over $X$ is an equivalence relation. We denote by $\mathcal{N}_G(X)$ the set of bordism classes of $n$-dimensional singular $G$-manifolds over $X$. The sets becomes an abelian group under disjoint union.

Proposition 1.6. $\mathcal{N}_G(X)$ is an equivariant homology theory, i.e. it satisfies the following:

1. Functorial in continuous $G$-maps.
2. $G$-equivariant homotopy invariant.
3. Takes $G$-weak equivalences to isomorphisms.

Construction 1.7. There is a distinguished class $d_{G,V} \in \tilde{\mathcal{N}}_G(S^V)$ for a $G$-representation $V$. Stereographic projection is a $G$-equivariant map

$$\Pi_V : S(\mathbb{R} \oplus V) \to S^V, \quad (x, v) \mapsto \frac{v}{1 - x}.$$

We define a reduced $G$-bordism class over $S^V$ by

$$d_{G,V} = [S(\mathbb{R} \oplus V), \Pi_V] \in \tilde{\mathcal{N}}_G(S^V).$$

Proposition 1.8. $d_{G,V} \wedge d_{G,W} = d_{G,V \oplus W} \in \tilde{\mathcal{N}}_{S^V \oplus S^W}$.

If $G$ acts trivially on $V$ and $X$ is a cofibrant based $G$-space, then the exterior product map with $d_{G,V}$ is an isomorphism:

$$\wedge d_{G,V} : \tilde{\mathcal{N}}_n(X) \to \tilde{\mathcal{N}}_{n+|V|}(X \wedge S^V).$$

Construction 1.9. To every smooth closed $G$-manifold $M$, we associate a normal class $\langle M \rangle \in \mathcal{M}_{-m}$. This class is the geometric input for the Thom-Pontryagin map to equivariant $mO$-homology. If $\dim M = m$, then the class lives in the summand $\mathcal{M}_{-m}$ of $\mathcal{M}$.

By Mostow-Palais embedding theorem, there is a $G$-equivariant embedding $i : M \hookrightarrow V$ for some $G$-representation $V$. Without loss of generality, assume $V$ is
a sub-representation of the chosen complete $G$-universe $\mathcal{U}_G$. Define $\nu$ to be the normal bundle of the embedding, where the metric is provided by the inner product on $V$. We can also assume, the embedding is wide in the sense that the exponential map $(x, v) \mapsto i(x) + v$ on the unit disk bundle $D(\nu)$ of $\nu$ is a $G$-embedding into a tubular neighborhood of $M$. This determines a $G$-equivariant Thom-Pontryagin map

$$c_M : S^V \longrightarrow Th(Gr(V)) \wedge M_+ = MGr(V) \wedge M_+$$

by sending points outside the tubular neighborhood to the base point and

$$c_M(i(x) + v) = \left(\frac{v}{1 - |v|}, \nu_x\right) \wedge x.$$ 

The normal class is the homotopy class of the collapse map $c_M$.

**Proposition 1.10.** The normal class does not depend on the choice of a wide embedding.

**Construction 1.11.** Equivariant Thom-Pontryagin construction:

$$\Theta^G = \Theta^G(X) : \mathcal{N}_*_G^*(X) \longrightarrow m\mathcal{O}_G^*(X).$$

Let $(M, h)$ be an $m$-dimensional singular $G$-manifold over a based $G$-space $X$. All the geometry is encoded in the normal class $\langle M \rangle \in MGr^G(M_+)$. We define

$$\Theta^G[M, h] = (b \wedge h)_\ast \langle M \rangle : p^G_\ast(\sigma^m) \in m\mathcal{O}_m^G(X),$$

where $b : MGr \rightarrow m\mathcal{O}$ is a map of ring spectra whose value at an inner product space $V$ is

$$b(V) : MGr(V) \rightarrow m\mathcal{O}(V), \quad (x, L) \mapsto ((x, 0), (L \oplus 0)),$$

$\sigma \in \pi^G_1(m\mathcal{O}[1])$ is periodicity class, inverse to $t \in \pi^G_{-1}(m\mathcal{O}[i-1])$ represented by

$$(0, \{0\}) \in Th(Gr_0(\mathbb{R} \oplus \mathbb{R}^{\infty})) = m\mathcal{O}[i-1](\mathbb{R}),$$

and $p_G : G \rightarrow e$ is the projection map that induces a map $p^G_\ast : \pi^G_\ast(-) \rightarrow \pi^G_\ast(-)$.

**Proposition 1.12.** The class $\Theta^G[M, h] \in m\mathcal{O}_m^G(X)$ only depends on the bordism class of the singular $G$-manifold $(M, h)$.

**Example 1.13.** $\Theta^G(d_G, V) = \tau_{G, V} \in m\mathcal{O}_m^G(S^V)$ is the shifted inverse Thom class in $m\mathcal{O}$.

**Theorem 1.14.** $\Theta^G$ is a transformation of equivariant homology theories and compatible with homomorphisms of compact Lie groups.

**Theorem 1.15** (Wasserman). Let $G$ be a compact Lie group that is isomorphic to a product of finite group and a torus. Then for every cofibrant $G$-space $X$, the Thom-Pontryagin map

$$\Theta^G(X) : \mathcal{N}_*_G^*(X) \longrightarrow m\mathcal{O}_G^*(X_+)$$

is an isomorphism.
Construction 1.16. We define stable equivariant bordism groups $\tilde{\Omega}^G_{*,*}(X)$ of a based $G$-space $X$ as the localization of $\tilde{\Omega}^G_{*,*}(X)$ by formally inverting all classes $d_{G,V}$. That is
$$\tilde{\Omega}^G_{*,}(X) = \text{colim}_{V \in s(U_G)} \tilde{\Omega}^G_{*,*}(X \wedge S^V),$$
where $s(U_G)$ is the poset of finite dimensional $G$-representations in the $G$-universe $U_G$ and for $V \subseteq W$ the structure map in the colimit is the multiplication
$$\tilde{\Omega}^G_{*,*}(X \wedge S^V) \to \tilde{\Omega}^G_{*,*}(X \wedge S^W).$$

As the Thom-Pontryagin construction takes $d_{G,V}$ to the shifted inverse Thom class $\tau_{G,V}$, the following diagram commutes

$$\begin{align*}
\tilde{\Omega}^G_{*,*}(X) & \xrightarrow{\Theta^G} m\Omega^G_{*,*}(X) \\
- \wedge d_{G,V} & \downarrow \quad \downarrow -\tau_{G,V} \\
\tilde{\Omega}^G_{*,*}(X \wedge S^V) & \xrightarrow{\Theta^G} m\Omega^G_{*,*}(X \wedge S^V).
\end{align*}$$

The colimit of this diagram assembles into a natural transformation:
$$\Theta^G : \tilde{\Omega}^G_{*,*}(X) \to m\Omega^G_{*,*}(X)[1/\tau].$$

Theorem 1.17. For every compact Lie group $G$ and every cofibrant based $G$-space $X$, then map
$$\Theta^G(X) : \tilde{\Omega}^G_{*,*}(X) \to m\Omega^G_{*,*}(X)[1/\tau]$$
is an isomorphism of graded abelian groups.

Corollary 1.18. For a cofibrant based $G$-space, there are natural isomorphisms:
$$\tilde{\Omega}^G_{*,*}(X) \xrightarrow{\Theta^G} m\Omega^G_{*,*}(X)[1/\tau] \xrightarrow{\sim} m\Omega^G_{*,*}(X).$$

2 Equivariant complex cobordism spectra

2.1 Complex cobordism and formal groups

Definition 2.1. A cohomology theory is called complex oriented if it is multiplicative and it satisfies Thom isomorphism for (almost) complex vector bundles.

Proposition 2.2. Let $E$ be a complex oriented cohomology theory, then

1. $E^*(\mathbb{CP}^\infty) \cong E_u[t]$ where $t \in E^2(\mathbb{CP}^\infty)$ is the first Chern class of the tautological line bundle $\xi$ over $\mathbb{CP}^\infty$. 

2. Let $p_i : \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ be the projection map of the $i$-th component for $i = 1, 2$. Then $E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \cong E_u[t_1, t_2]$, where $t_i = p_i^*c_1(\xi)$. 

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3. The tensor product of line bundles over $\mathbb{C}P^2$ induces a $E_0$-formal group structure on $spf E(\mathbb{C}P^2)$. Denote this formal group associated to a complex-oriented cohomology theory $E$ by $\hat{G}_E$.

4. $E(S^{2k})$ can be identified $\omega^k$, the $k$-th tensor power of the sheaf of invariant differentials on $\hat{G}_E$.

**Example 2.3.** Here are two examples of complex oriented cohomology theories and their associated formal groups:

1. For ordinary cohomology theory, $\hat{G}_H \approx \hat{G}_a$ is the additive formal group.
2. For complex $K$-theory, $\hat{G}_K \approx \hat{G}_m$ is the multiplicative formal group.

**Theorem 2.4 (Quillen).** The formal group associated to periodic complex cobordism $MUP$ is the universal formal group. More precisely, the pair

$$(MU_*, MU_*(MU)) = (MUP_0, MUP_0(MUP))$$

classifies formal groups and strict isomorphisms between formal groups.

2.2 Real bordism

Let $\rho_2$ be the real regular representation of $C_2$.

**Construction 2.5.** We construct the real cobordism spectrum $MUR$. It is a $C_2$-equivariant commutative ring admitting a canonical homotopy presentation

$$MUR \approx \operatorname{holim} S^{-C_n} \wedge MU(n) \approx \operatorname{holim} S^{-\rho_2} \wedge MU(n).$$

We will first construct a commutative real algebra $MUR \in CAlg(Sp_R)$ and apply the Quillen equivalence:

$$i^! : Sp_R \leftrightarrow Sp^{C_2} : i^* .$$

We define $MUR$ to be the spectrum $i_! MUR'$, where $MUR' \rightarrow MUR$ is a cofibrant commutative algebra approximation. Elements in this construction are described below:

**Definition 2.6.** The category $I_C$ is the topological category whose objects are finite dimensional Hermitian vector spaces and whose morphism space is the Thom space

$$I_C(A, B) = Th(U(A, B); B - A),$$

where $U(A, B)$ is the Stiefel manifold of unitary embeddings $A \hookrightarrow B$ and $B - A$ is the orthogonal complement of $A$ in $B$ under the embedding.

The category $I_R$ is the $C_2$-equivariant topological category whose objects are finite dimensional orthogonal real vector spaces and whose morphism space is the Thom space

$$I_R(V, W) = I_C(V_C, W_C),$$

with $C_2$ acting by complex conjugation.
Definition 2.7. The category $\text{Sp}_C$ of complex spectra is the topological category of (continuous) functors $I_C \to T$.

The category $\text{Sp}_R$ of real spectra is the topological category of $C_2$-enriched functors $I_R \to I_{C_2}$ and equivariant natural transformations.

Let $i : I_R \to I_{C_2}$ be the functor sending $V$ to $V \otimes \rho_2$. The restriction functor $i^* : \text{Sp}_{C_2} \to \text{Sp}_R$ has both a left and right adjoint denoted by $i_!$ and $i_*$. $i_!$ sends $S^V$ to $S^V\rho_2$.

The restriction functor $i_* : \text{Sp}_C \to \text{Sp}_R$ has an adjoint denoted by $i^!$.

We define the real spectrum $MU_R$ by sending $V \in I_R$ to $MU_{C_2}^p V \cup \text{Th}_{C_2}^p BU_0$, with $C_2$ acting by complex conjugation. $MU_R \in \text{CAlg}(\text{Sp}_R)$ as the functor is a lax symmetric monoidal if we use Segal’s construction of $BU(V_C)$.

Proposition 2.8.

1. The non-equivariant spectrum underlying $MU_R$ is the usual complex cobordism spectrum $MU$.

2. There is an equivalence $\Phi^{C_2} MU_R \simeq MO$.

We now describe the relations between $MU_R$, real orientations and formal groups. Consider $\mathbb{C}P^n$ and $\mathbb{C}P^\infty$ as pointed $C_2$-spaces under complex conjugation, with $\mathbb{C}P^0$ the base point. The fixed point spaces are $\mathbb{R}P^n$ and $\mathbb{R}P^\infty$, and they are homeomorphisms $\mathbb{C}P^n/\mathbb{C}P^{n-1} \simeq S^{n\rho_2}$. In particular $\mathbb{C}P^1 \simeq S^{2\rho_2}$.

Definition 2.9 (Araki). Let $E$ be $C_2$-equivariant homotopy commutative ring spectrum. A real orientation of $E$ is a class $\pi \in E_{C_2}^n(\mathbb{C}P^n)$ whose restriction to $E_{C_2}^n(\mathbb{C}P^1) = E_{C_2}^0(S^0) \simeq E_{C_2}^0(pt)$ is a unit. A real oriented spectrum is a $C_2$-equivariant ring spectrum $E$ equipped with a real orientation.

Example 2.10. The zero section $\mathbb{C}P^\infty \to MU(1)$ is an equivariant equivalence and defines a real orientation $\pi \in MU_{C_2}^1(\mathbb{C}P^\infty)$, making $MU_R$ into a real oriented spectrum.

Example 2.11. If $(X, x_H)$ and $(E, x_E)$ are two real oriented spectra, then $H \wedge E$ has two real orientations given by $x_H \otimes 1$ and $1 \otimes x_E$.

Theorem 2.12 (Araki). Let $E$ be a real oriented cohomology theory, then there are isomorphisms

$$E^*(\mathbb{C}P^\infty) \simeq E^*[\pi],$$

$$E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \simeq E^*[\pi \otimes 1, 1 \otimes \pi].$$

It follows the tensor product map $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ defines a formal group law over $\mathbb{C}P^\infty$. A real orientation $\pi$ corresponds to a coordinate the corresponding formal group.
If $(E, x_E)$ is a real oriented spectrum, then $E \wedge MU_R$ has two orientations $x_E = x_E \otimes 1$ and $x_R = 1 \otimes x$. These two series are related by a power series

$$x_R = \sum b_i x_E^{i+1},$$

that defines classes $b_i = b_i^E \in \pi_1^C E \wedge MU_R$.

This power series is an isomorphism of formal group laws $F_E$ to $F_R$ over $\pi_2 C E \wedge MU_R$, where $F_E$ and $F_R$ are formal groups associated to $(E, x_E)$ and $(MU_R, x_R)$, respectively.

**Theorem 2.13 (Araki).** The map

$$E_*[b_1, b_2, \cdots] \to \pi_*^C E \wedge MU_R$$

is an isomorphism.

Passing to geometric fixed points

$$x: \mathbb{CP}^\infty \to \Sigma^2 MU_R \xrightarrow{\text{geom fixed pt}} a: \mathbb{RP}^\infty \simeq MO(1) \to \Sigma MO$$

defines the MO Euler class of the tautological line bundle. Like $MU_*$, Quillen shows that the multiplication $\mathbb{RP}^\infty \times \mathbb{RP}^\infty \to \mathbb{RP}^\infty$ induces a formal group law over $MO_*$ that is universal formal group law $F$ over a ring of characteristic 2 such that $[2]_F = 0$.

Let $e \in H^1(\mathbb{RP}^\infty; \mathbb{Z}/2)$ be the $\mathbb{HZ}/2$ Euler class (Stiefel-Whitney class) of the tautological line bundle. Over $\pi_*^C(\mathbb{HZ}/2 \wedge MO)$, the classes $e$ and $a$ are related by a power series

$$e = \ell(a) = a + \sum \alpha_n a^{n+1}.$$

**Lemma 2.14.** The composite series

$$\left(a + \sum \alpha_{2^k - 1} a^{2^k}\right)^{-1} \circ \ell(a) = a + \sum_{j>0} h_j a^{j+1}$$

has coefficients in $\pi_* MO$. The classes $h_{2^k-1} = 0$ and the remaining $h_j$ are polynomial generators for the unoriented cobordism ring:

$$\pi_* MO = \mathbb{Z}/2[h_j \mid j \neq 2^k - 1].$$

Let $G = C_2^\ast$ and localize all spectra at the prime 2. Write $g = |G|$ and let $\gamma \in G$ be a fixed generator.

**Definition 2.15.** $MU^G := N_{C_2}^G MU_R$

For $H \subset G$, the unit of the restriction-norm adjunction gives a canonical commutative algebra map

$$MU^H \to i^H_* MU^G.$$

Write $i^H_*$ for $i^G_*$. 

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2.3 Universal properties of real bordism

Let \( R_\ast \) be a graded ring and \( F(x, y) \in R_\ast[[x, y]] \) be a homogeneous formal group (\( \deg x = \deg y = -2 \)). Let \( c : R_\ast \to R_\ast \) be a graded ring homomorphism such that \( c_{2n} : R_{2n} \to R_{2n} \) is multiplication by \((-1)^n\). Define \( F^c = c^* F \), we have

\[
F^c(x, y) = -F(-x, -y).
\]

c induces strict isomorphisms \( F \xrightarrow{\sim} F^c \) and \( F^c \xrightarrow{\sim} F \) by \( c(x) = -[-1]_F(x) \). This is called the conjugate action on \( F \).

**Proposition 2.16.** \([HHR, Example 11.27]\) \( MU_\ast \) is universal in the sense that \( MU_\ast \to R_\ast \) classifying a homogeneous formal group law is \( C_2 \)-equivariant for any choice of conjugation action.

The real orientation \( i^* MU_\ast \to MU(\ast) \) for \( G = C_2 \) induces a formal group law \( F \) with a \( G \)-action that extends the conjugation action on by \( C_2 \subseteq G \).

**Proposition 2.17.** \([HHR, Proposition 11.28]\) This pair \( (MU(\ast), F) \) is universal in the sense that

\[
\text{Hom}_{\text{gr}} \left( \pi^u_\ast \left( MU(\ast) \right), R_\ast \right) \simeq \left\{ \text{Formal groups over } R_\ast \text{ with a } G \text{-action extending the conjugation action by } C_2 \subseteq G \right\}
\]

**References**
