The goal of these two talks is to prove the detection theorem:

**Theorem 1.** [2, Theorem 11.1] If \( \theta_j \in \pi_{2j+1-2}S^0 \) is an element of Kervaire invariant 1, and \( j > 2 \), then the image of \( \theta_j \) in \( \pi_{2j+1-2}\Omega \) is non-zero.

We will use Adams-Novikov spectral sequence (ANSS) to study \( \theta_j \in \pi_{2j+1-2}S^0 \) and its image in \( \pi_{2j+1-2}\Omega \). This will reduce the problem to an algebraic one on the \( E_2 \) page. The algebraic problem can be reduced to an easier one via a construction using formal \( A \)-modules. The goal of this talk is to briefly introduce ANSS and formal \( A \)-modules.

## 1 Adams-Novikov Spectral Sequence

### 1.1 Construction

Set up: let \( E \) be an associative ring spectrum (homotopy commutative) and assume that \( E_\ast E \) is flat over \( E_\ast \).

**Theorem 2.** [1, Theorem 15.1] Given a spectrum \( X \), we have the \( E \)-based Adams-Novikov spectral sequence

\[
E_2^{s,t} = \text{Ext}^s_{E_\ast, E}(E_\ast, E_{t+1}X) \Rightarrow \pi_{t-s}X_E^s.
\]

The construction follows from the cosimplicial resolution:

\[ X_\bullet : X \to E \wedge X \to E \wedge E \wedge X \to \cdots. \]

The total space \( \text{Tot}(X_\bullet) \) is \( X_E^s \) and the ANSS is the Bousfield-Kan spectral sequence.

### 1.2 \( E_2 \) page

**Definition 3.** [5, A 1.1.1] A Hopf algebroid over a commutative ring \( K \) is a cogroupoid in the category of (graded or bigraded) commutative \( K \)-algebras, i.e., a pair \( (A, \Gamma) \) of commutative \( K \)-algebra with structure maps such that for any commutative \( K \)-algebra \( B \), the sets \( \text{Hom}(A, B) \) and \( \text{Hom}(\Gamma, B) \) are the objects and morphisms of a groupoid.
The pair \((E_*, E, E)\) is a Hopf algebroid. Here is the structure.

\[
E \to E \wedge E \to E \wedge E \wedge E
\]

where the first part has three maps: left unit, right unit, multiplication; the second map is \(\text{id} \wedge \text{unit} \wedge \text{id}\). We can identify \(E \wedge E \wedge E\) with \((E \wedge E) \wedge (E \wedge E)\) (\(E\) is associative). Applying \(\pi_*\), the second part gives the coproduct \((E_*, E)\) is flat over \(E_*\)

\[
E_*E \to E_*E \otimes E_*E.
\]

Similarly, \(E_*X\) is an \(E_*E\) comodule.

In general, if \((A, \Gamma)\) is a Hopf algebroid and \(M\) is a \(\Gamma\) comodule, we have the cobar resolution

\[
C^*_{\Gamma} : M \to \Gamma \otimes_A M \to \Gamma \otimes_A^2 M \to \cdots.
\]

By definition, the cohomology \(H^*(C^*_\Gamma)\) is \(\text{Ext}^*_{E_*E}(E_*, M)\). One can identify the \(E_2\) page from this.

### 1.3 Examples

**Example 4.** The classical Adams spectral sequence Let \(E\) be \(HF_2\), \(X\) be the sphere spectrum \(S\). Then \(S \wedge HF_2\) is the 2-completed sphere \(S_2^\wedge\) and

\[
E_*E = A = \mathbb{F}_2[\xi_1, \xi_2, \cdots], \quad |\xi_i| = 2^i - 1
\]

is the dual Steenrod algebra.

Here are some interesting elements on the \(E_2\) page \(\text{Ext}_A(\mathbb{F}_2, \mathbb{F}_2)\). In this case, the cobar resolution to compute the \(E_2\)-page is

\[
\mathbb{F}_2 \to A \to A \otimes A \to \cdots.
\]

In degree 1, we will write \(\xi_1 \in A\) as \([\xi_1]\). In degree 2, we will write \(\xi_1 \otimes \xi_1 \in A \otimes A\) as \([\xi_1|\xi_1]\). You can see the pattern.

\[
[\xi_1^2] = h_0 \in \text{Ext}^1 \Rightarrow \text{Hopf invariant 1 elements}.
\]

For example, \(h_0\) converges to 2, \(h_1\) converges to \(\eta\), but from \(h_4\), they do not survive in the homotopy.

\[
d_2 h_i = h_0 h_{i-1}.
\]

\[
[\xi_1^2|\xi_1^2] = h_1^2 \in \text{Ext}^2 \Rightarrow \text{Kervaire invariant 1 elements}.
\]

**Example 5.** The Adams-Novikov spectral sequence Let \(E\) be the \(p\) primary Brown-Peterson spectrum \(BP\), \(X\) be the sphere spectrum \(S\). Then \(S_0^\wedge\) is the \(p\)-local sphere \(S^{(2)}\) and

\[
BP_* = \mathbb{Z}_p[v_1, v_2, \cdots], \quad |v_i| = 2(p^i - 1),
\]

\[
BP_*BP = BP_*[t_1, t_2, \cdots], \quad |t_i| = 2(p^i - 1).
\]
In this case, the cobar resolution to compute the $E_2$-page is

$$BP_* \to BP_*BP \otimes BP_* \to BP_*BP \otimes BP_*BP \otimes BP_* \to \cdots.$$ 

For elements in degree 1, for example $t_1 \otimes 1 \in BP_*BP \otimes BP_*$, we use the notation $[t_1]1$, sometimes we omit the 1 and write $[t_1]$ for this class. For elements in degree 1, for example $t_1^2 \otimes t_1 \otimes v_1 \in BP_*BP \otimes BP_*BP \otimes BP_*$, we use the notation $[t_1^2][t_1]v_1$ and you can see the pattern.

### 1.4 Thom Reduction

There is a map from ANSS to ASS induced by the map of Hopf Algebroid

$$(BP_*, BP_*BP) \to (HF_2, A)$$

where $BP_* \to HF_2$ is the quotient map by $(2, v_1, \cdots)$ and $BP_*BP \to A$ sends $t_i$ to $\bar{\xi}_i^2$.

#### Example 6.

Under Thom reduction, $[t_1]$ goes to $[\bar{\xi}_1] = [\xi_1]$.

### 1.5 Greek letter elements

The structure map in $(BP_*, BP_*BP)$ is much more complicated. One can refer to [5, Theorem A2.1.27] for the explicit formulas if one would like to try the computation by hands. To have a better understanding of the $E_2$ page, we introduce the greek letter elements. We will write $\text{Ext}_{BP_*BP}(BP_*, M)$ as $\text{Ext}(M)$ for short.

#### Definition 7. [5, Section 1.3]

Denote the idea $(p, v_1, \cdots, v_{n-1}) \subset BP_*$ by $I_n$. The short exact sequence

$$0 \to BP_*/I_n^\infty \to BP_*/I_n^\infty[v_{n-1}] \to BP_*/I_n^{\infty-1} \to 0$$

induces a long exact sequence in Ext groups. Denote the boundary map by

$$\delta_n: \text{Ext}^s(BP_*/I_n^\infty) \to \text{Ext}^{s+1}(BP_*/I_n^{\infty-1}).$$

Suppose that $x = v_n^h/(p^h, v_1^h), \cdots, v_{n-1}^h + \cdots$ (we only write the leading term) is an element in $\text{Ext}^0(BP_*/I_n^\infty)$, then

$$\alpha^{(n)}_{i_n/i_{n-1}, \cdots, i_0} := \delta_1 \circ \delta_2 \circ \cdots \circ \delta_n x$$

where $\alpha^{(n)}$ is the $n$th Greek letter. When $i_0 = 1$, we omit it from the notation.

#### Example 8.

$\alpha$ family and $\beta$ family.

$$\beta_{i/j} \in \text{Ext}^{2\beta_i-2}(BP_*, BP_*).$$

#### Fact 9.

In the bidegree of $\theta_{i,j}$, there is only one class $h^2_i$ in the classical Adams spectral sequence. However, in the ANSS, there are more than one elements. For example, in the bidegree of $\theta_4$, there are $\beta_{9/8}, \beta_{6/2}$ and $\alpha_{102 \cdots -1}$ ([6, Section 7.1]).
The first two lines of the ANSS have been studied in [4]. We follow notations in [6] to describe classes in ANSS in the bidegree of $\theta_j$ as follows:

$$\beta_{c(j,k)/2^{j-1-2k}},$$

$$\alpha_{c(j,k)/2^{j-1}},$$

where $0 \leq k < j$ and $c(j, k) = 2^{j-1-2k}(1 + 2^{2k+1})/3$.

2 Formal $A$-modules

Follow the notations in [2, Section 11.2]. Let $A$ and $R$ be commutative rings, and $e: A \to R$ a ring homomorphism.

Definition 10. [5, A2.1.1] A (commutative 1-dimensional) formal group law over $R$ is a power series $F(x, y) \in R[[x, y]]$ satisfying

1. $F(x, 0) = F(0, x) = x$,
2. $F(x, y) = F(y, x)$,
3. $F(x, F(y, z)) = F(F(x, y), z)$.

Definition 11. [5, A2.1.5] Let $F$ and $G$ be formal group laws. A homomorphism from $F$ to $G$ is a power series $f(x) \in R[[x]]$ with constant term 0 such that

$$f(F(x, y)) = G(f(x), f(y)).$$

It is an isomorphism if it is invertible, i.e., if $f'(0)$ (the coefficient of $x$) is a unit in $R$, and a strict isomorphism if $f'(0) = 1$. A strict isomorphism from $F$ to the addition formal group law $x + y$ is a logarithm for $F$, denoted by $\log_F(x)$.

We have a ring homomorphism

$$\mathbb{Z} \to \text{End}(F),$$

$$n \to [n](x)$$

where $[1](x) = x$ and $[n + 1](x) = F(x, [n](x))$.

Definition 12. A formal $A$-module over $R$ is a formal group law $F$ over $R$, equipped with a ring homomorphism

$$A \to \text{End}(F),$$

$$a \to [a](x)$$

with the property that $[a]'(0) = e(a)$. 
We are interested in the case $A = \mathbb{Z}_2[\zeta]$ where $\zeta$ is a primitive $8^{th}$ root of unity and $R_s = A[u, u^{-1}]$ where $|u| = 2$. The maximal ideal of $A$ is generated by $\pi = \zeta - 1$. $A$ is a discrete valuation ring. The valuation can be given by the divisibility of $\pi$. For example, 2 is $\pi^2$-unit in $A$.

Given a power series $f(x) \in A[[x]]$ such that
\[
 f(x) = \pi x \mod(x^2) \\
 f(x) = x^2 \mod(\pi),
\]
Lubin and Tate’s work $[3]$ constructed a formal $A$-module $F_f$ over $A$ (unique up to isomorphism) such that
\[
 [\pi](x) = f(x).
\]
For $a \in A$, write
\[
 [a](x) = a_d x^d + \cdots \mod(\pi)
\]
with $0 \neq a_d \in A/(\pi)$ One can check that the function $v(a) = \log_q(d)$ defines a valuation on $A$. For example, $v(\pi) = 1$, $v(2) = 4$. We can define a homogeneous formal group law over a graded ring by setting $|x| = |y| = -2$. From a formal group law $F_f$, we can define a homogeneous formal group law $F$ over $R_s$ by
\[
 uF(x, y) = F_f(ux, uy).
\]

3 Group actions and the ANSS $[2, 11.3.2, 11.3.3]$

Let $\mathcal{M}_{FG}$ be the category of pairs $(R, F)$, with $F$ a formal group law over a commutative ring $R$, and in which a morphism
\[
 (f, \psi): (R_1, F_1) \to (R_2, F_2)
\]
consists of a ring homomorphism $f: R_1 \to R_2$, and an isomorphism of formal group laws $\psi: F_2 \cong f_*F_1$. A (left) action of a group on $(R, F)$ is a map of monoids
\[
 G \to \mathcal{M}_{FG}((R, F), (R, F)).
\]

We define a trivial $C_8 = \langle \gamma \rangle$ action on $A$. Then $C_8$ acts on the pair $(A, F_f)$ by
\[
 f_\gamma: A \xrightarrow{\text{id}} A, \\
 \psi_\gamma = [\zeta](x): F_f \to f_\gamma F_f = F_f.
\]
The $C_8 = \langle \gamma \rangle$ action can be extended to $(R_s, F)$ by
\[
 \gamma u = \zeta u.
\]

Example 13. $[2, \text{Example 11.18}]$ Here is an example of $(R, F)$ with $G$ action. Suppose that $E$ is a complex oriented, homotopy commutative ring spectrum, and that a finite group $G$ acts on $E$ by homotopy multiplicative maps. Let $F$ denote the corresponding (homogeneous) formal group law over $\pi_*E$. Then the action of $G$ on $E^*(\mathbb{C}P^\infty)$ gives an action of $G$ on $(\pi_*E, F)$.  

One can associate a Hopf algebroid to a pair \((R, F)\) with \(G\) action. Let \(C(G, R_*)\) be the ring of maps (as set) from \(G\) to \(R_*\). (In our case, \(G = C_8\) and \(R_* = \mathbb{Z}_2[[u, u^{-1}]]\).) The pair \((R_*, C(G, R_*))\) is a Hopf algebroid. The structure maps are

\[
\eta_L : R_* \rightarrow C(G, R_*)
\]

sending \(r \in R_*\) to the constant function with value \(r\);

\[
\eta_R : R_* \rightarrow C(G, R_*)
\]

sending \(r \in R_*\) to the function \(g \rightarrow g \cdot r\);

\[
\Delta : C(G; R_*) \rightarrow C(G; R_*) \otimes_{R_*} C(G; R_*)
\]

dual to multiplication in \(G\), and the isomorphism

\[
C(G; R_*) \otimes_{R_*} C(G; R_*) \overset{\sim}{\rightarrow} C(G \times G; R_*)
\]

given by setting

\[
(f_1 \otimes f_2)(g_1, g_2) = f_1(g_1) \cdot g_1 f_2(g_2).
\]

Moreover, there is a map of Hopf algebroids

\[
(MU_*, MU, MU) \rightarrow (R_*, C(G, R_*))
\]

where the map \(MU_* \rightarrow R_*\) classifies the formal group law \(F\), and the map \(MU, MU \rightarrow C(G, R_*)\) is defined by declaring the composition

\[
MU_* MU \rightarrow C(G, R_*) \overset{ev}{\rightarrow} R_*
\]

to be the map classifying the strict isomorphism

\[
[g](x) : F \rightarrow g^* F.
\]

(Here we use the fact that \(MU_*, MU\) represents strict isomorphism between formal group laws. A map \(MU_*, MU \rightarrow R\) is equivalent to a strict isomorphism between \(F_1\) and \(F_2\).)

The map \([1]\) induces a map

\[
\text{Ext}^{s,t}_{MU, MU}(MU_*, MU_*) \rightarrow H^s(G; R_*)
\]

(2)

When the \(G\)-action on \((R_*, F)\) arises, as in Example \([13]\) from an action of \(G\) on a complex oriented homotopy commutative ring spectrum \(E\), the map \([2]\) is the \(E_2\)-term of a map of spectral sequences abutting to the homomorphism \(\pi_* S^0 \rightarrow \pi_* E^{hG}\) (see details in \([2] 11.3.3\)).
References


