Little n-cubes, E_n & E_∞

Recall

Operad: $\{\mathcal{C}(j)_{\circlearrowleft\Sigma_j}, 1, \gamma\}$ + assoc. unital, Σ_j-equivariance

$\mathcal{C}$-algebra: $\{\mathcal{C}(j) \times X^j \longrightarrow X\}$

Def An A_∞-operad is a Σ-free operad $\mathcal{C}$ w/ a morphism $\mathcal{C} \longrightarrow \mathcal{M}$ s.t. the level-wise map $\mathcal{C}(n) \longrightarrow \mathcal{M}(n)$ is a Σ_n-equiv. map. (local Σ-equiv)

Def An E_∞-operad is a Σ-free operad $\mathcal{C}$ s.t. $\mathcal{C} \longrightarrow \mathcal{N}$ has level-wise $\mathcal{C}(n) \longrightarrow \mathcal{N}(n)$ homotopy equivalence. (local equiv.)

Rk $\mathcal{M}$, since free, is itself an A_∞-operad, but $\mathcal{N}$ is not an E_∞-operad!

E.g. ① Barrat-Eccles operad.

$\mathcal{E}(n) := E\Sigma_n$, Σ-free, contractible.

$\gamma: \mathcal{E}(k) \times \mathcal{E}(j_1) \times \cdots \times \mathcal{E}(j_k) \longrightarrow \mathcal{E}(j)$ induced by $\Sigma_k \times \Sigma_{j_1} \times \cdots \times \Sigma_{j_k} \to \Sigma_j$

② Linear isometry operad, Steiner operad. (Yu's talk)

③ Little cube operad $\mathcal{C}_\infty$, defined below.

E_∞-algebras are A_∞-algebras

Def Given $\mathcal{C}, \mathcal{C}'$, define $\mathcal{C} \times \mathcal{C}'$ by $\mathcal{C} \times \mathcal{C}'(j) := \mathcal{C}(j) \times \mathcal{C}'(j)$ &

① $\gamma \times \gamma'(c \times c', d_1 \times d_1', \cdots d_k \times d_k') = \gamma(c; d_1, \cdots d_k) \times \gamma'(c'; d_1', \cdots d_k')$

② $1 = 1_{\mathcal{C}} \times 1_{\mathcal{C}'} \in \mathcal{C}(1) \times \mathcal{C}'(1)$

③ $(c \times c')\sigma = c\sigma \times c'\sigma$

Prop Let $\mathcal{C}$ be an E_∞-operad, $\mathcal{C}'$ be any Σ-free operad. Then

$$\mathcal{C}\times\mathcal{C}' \longrightarrow \mathcal{C}'$$

is a levelwise htpy equiv. In particular, $\mathcal{C}\times\mathcal{M} \longrightarrow \mathcal{M}$ is a level-wise Σ_j-equiv. $\Longrightarrow$ $\mathcal{C}\times\mathcal{M}$ is an A_∞-op. $\Longrightarrow$ (E_∞-alg is A_∞-alg).

Slogan: E_n interpolates between A_∞ & E_∞

Little n-cube operads

Def (Geo. of Iter. Loop sp)

I^n = unit n-cube $\quad J^n = \mathrm{Int}(I^n)$ = interior of I^n.

An (open) little n-cube is a linear embedding $f: J^n \hookrightarrow J^n$, w/ parallel axes; thus $f = f_1\times\cdots\times f_n$ for each $f_i: J \hookrightarrow J$ a linear fcn.

Define $\mathcal{C}_n(j) := \{\langle c_1,\cdots c_j\rangle \mid c_i$ are n-cubes, im c_i are pairwise disjoint$\}$.

Given $\mathcal{C}_n(j)$ the subspace topology from $\mathrm{Hom}_{Top}(\coprod_j J^n, J^n)$. Write $\mathcal{C}_n(0) = \{\langle\,\rangle\}$ regarded as the unique "embedding" of ϕ in J^n.

The requisite data are defined by

① $\gamma(c;d_1,\cdots d_k) = c\circ(d_1+\cdots+d_k): \coprod_{j_1} J^n \amalg\cdots\amalg \coprod_{j_k} J^n \longrightarrow J^n$, $c\in\mathcal{C}_n(k)$, $d_s\in\mathcal{C}_n(j_s)$

② $1\in\mathcal{C}_n(1)$ is $\mathrm{id}: J\to J$ (draw pictures)

③ $\langle c_1,\cdots c_j\rangle\sigma = \langle c_{\sigma(1)},\cdots c_{\sigma(j)}\rangle$ for $\sigma\in\Sigma_j$, & thus the action is free.

Define a morphism of operads $\sigma_n: \mathcal{C}_n \longrightarrow \mathcal{C}_{n+1}$ by

$$\sigma_{n,j}: \mathcal{C}_n(j) \longrightarrow \mathcal{C}_{n+1}(j)$$

$$\langle c_1,\cdots c_j\rangle \longmapsto \langle c_1\times \mathrm{id}_J,\cdots c_j\times\mathrm{id}_J\rangle$$

$\rightsquigarrow$ each $\sigma_{n,j}$ is an inclusion $\rightsquigarrow$ $\mathcal{C}_\infty := \varinjlim_n \mathcal{C}_n$

Rk There is another operad called the little n-disk operad. The ideas are similar except replacing all cubes by disks. The two operads are htpy equiv to each other.

The topology of $\mathscr{C}_n$ can be described using configuration spaces:

Def Define the j^{th} configuration space $F(M;j)$ of M by

$$F(M;j) = \{\langle x_1, \cdots x_j \rangle \mid x_r \in M,\ x_r \neq x_s \text{ if } r \neq s\} \subseteq M^j$$

Let Σ_j acts on $F(M;j)$ by $\langle x_1 \cdots x_j \rangle \sigma = \langle x_{\sigma(1)}, \cdots x_{\sigma(j)} \rangle$, which is free,

Some facts:

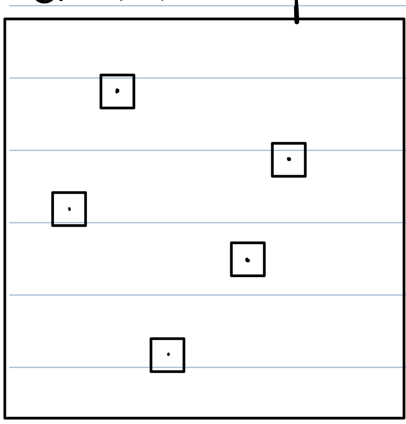
$F(J^\infty; j)$ is Σ_j-free & contractible, i.e. $E\Sigma_n$.

proof: Fadell-Neuwirth fibration + induction.

$F(J'; j)$ has Σ_j-components & each is contractible

proof: One component of $F(J';j)$ is $F_0 = \{\langle x_1, \cdots x_j \rangle \mid x_1 < \cdots < x_j\}$ & Σ_j permutes the x_i's. F_0 is the interior of a simplex, & thus contractible.

Thm For $1 \le n \le \infty$, $\mathscr{C}_n(j)$ is Σ_j-equivariantly equiv. to $F(J^n; j)$. Therefore, $\mathscr{C}_1$ is an A_∞-operad. $\mathscr{C}_n$ is locally $(n-2)$-connected Σ-free operad, & $\mathscr{C}_\infty$ is an E_∞-operad.



proof: Define a map $g: \mathscr{C}_n(j) \longrightarrow F(J^n; j)$ by

$\langle c_1, \cdots c_j \rangle \longmapsto \langle c_1(\gamma), \cdots c_j(\gamma) \rangle \qquad \gamma = (\frac{1}{2}, \cdots \frac{1}{2}) \in J^n$.

Conversely, define $f: F(J^n; j) \longrightarrow \mathscr{C}_n(j)$ by

$\langle b_1, \cdots b_n \rangle \longmapsto$ (cube centered at each b_i w/ equal & maximal diameter).

Then clearly $gf = 1$.

$fg \simeq 1$ because contracting/expanding squares is homotopied.

E_n/E_∞-algebras

Thm (Recognition Principle)

For $\forall\ 1 \leq n \leq \infty$, every n-fold loop space is a $\mathcal{C}_n$-space, & every connected $\mathcal{C}_n$-space has the weak htpy type of an n-fold loop space.

In the case of $n=\infty$

Thm If X is an E_∞-space & $\pi_0(X)$ is a group (extending the monoid structure on $\pi_0(X)$), then $X \simeq \Omega^\infty Z$ for some Sp Z.

i.e. Ω^∞ defines an equivalence between $\underset{\text{infinite loop spaces.}}{\underline{\text{connective spectra}}}$ & grouplike E_∞-spaces.

proof: later talks.

Def Naively, Y is an n-fold loop space if $\exists\ X$ s.t. $Y = \Omega^n X$ in Top_*.
However, $Y = \Omega^n X$ does not uniquely determine X, and we need to remember X in order to have a well-defined <u>category of n-fold loop spaces.</u> ($\mathcal{L}_n$)

Obj: $\{Y_i \mid 0 \leq i \leq n \text{ w/ } Y_i = \Omega Y_{i+1}\}$ or $\{Y_i \mid 0 \leq i \text{ w/ } Y_i = \Omega Y_{i+1}\}$

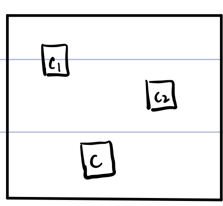
Mor: $\{g_i : Y_i \longrightarrow Y_i' \mid g_i = \Omega g_{i+1}\}$

for simplicity, denote an n-fold loop space by $\Omega^n X$.

We have a functor $U_n : \mathcal{L}_n \longrightarrow Top_*$ taking $\{Y_i\}$ to Y_0.

For $n = \infty$, $U_n = \Omega^\infty$

Thm The little n-cube operad $\mathcal{C}_n$ acts on $\Omega^n X$ naturally.



$$\theta_{n,j} : \mathcal{C}_n(j) \times (\Omega^n X)^j \longrightarrow \Omega^n X.$$

$$\theta_{n,j}(\langle c_1, \cdots c_j\rangle, (y_1, \cdots y_j))(v) \longmapsto \begin{cases} y_r(u) & \text{if } c_r(u) = v \\ * & \text{if } v \in \mathrm{Im}\, c \end{cases}$$

Towards the Approximation Thm:

Thm We have the adjunction $\Sigma^n \dashv \Omega^n$ $\rightsquigarrow$ a monad $\Omega^n\Sigma^n$.
$\rightsquigarrow$ get a map $\alpha_n : C_n X \xrightarrow{C_n(\eta)} C_n \Omega^n\Sigma^n X \xrightarrow{\theta} \Omega^n\Sigma^n X$
which is a morphism of monads. & the following diagram of functors commutes

$$\begin{tikzcd} & \mathcal{L}_n \arrow[dl, "V_n"'] \arrow[dr] & \\ \Omega^n\Sigma^n\text{-alg} \arrow[rr, "\alpha_n^*"] & & C_n\text{-alg} \end{tikzcd}$$

where given an n-fold loop space $\Omega^n X$, $\Omega^n\Sigma^n$ acts on $\Omega^n X$ by the counit $\Omega^n\Sigma^n\Omega^n X \xrightarrow{\Omega^n \varepsilon X} \Omega^n X$.

Approx Thm α_n is a weak htpy equiv. for all n. if X is connected.