The approximation theorem.

Recall. $\mathcal{M}(j) = \Sigma_j \rightsquigarrow MX$ James construction.

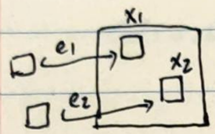
$\mathcal{N}(j) = *$ $\rightsquigarrow NX$ Infinite symmetric product.

$\mathcal{C}_n$: little n-cubes operad $\rightsquigarrow C_n$ monad

$\Sigma^n : \mathcal{T} \rightleftarrows \mathcal{T} : \Omega^n \rightsquigarrow$ monad $\Omega^n\Sigma^n$

$\mathcal{C}_1$ is A_∞.
$\mathcal{C}_n$ locally (re)-connected.
$\mathcal{C}_\infty$ is E_∞.

Map of monads $\alpha_n : C_n \to \Omega^n\Sigma^n$.



$\longmapsto$ $f : (I^n, \partial I^n) \to \Sigma^n X$

$$m \longmapsto \begin{cases} e_i^{-1}(m) \wedge x_i & m \in e_i(I^n) \\ * \end{cases}$$

$\sigma_n : \Omega^n\Sigma^n X \to \Omega^{n+1}\Sigma^{n+1}X \rightsquigarrow \Omega^\infty\Sigma^\infty X = \operatorname{colim} \Omega^n\Sigma^n X$

$\mathrm{Map}(S^n, \Sigma^n X) \to \mathrm{Map}(S^{n+1}, \Sigma^{n+1}X)$ $\quad \downarrow \alpha_\infty$

$\sigma'_{n+1,j} : \mathcal{C}_{n+1}(j) \to \mathcal{C}_n(j) \rightsquigarrow C_\infty X = \operatorname{colim} C_n X$

$f \longmapsto 1 \times f$

Theorem. $\alpha_n : C_n X \to \Omega^n\Sigma^n X$ is a weak homotopy equivalence if X is connected.

(weak group completion in general)

$M \to \Omega BM \simeq \Omega^n\Sigma^n X$ + Quillen
π_0 grp cplt, $\Rightarrow H_*(M)[\pi_0^{-1}] \cong H_*(\Omega^n\Sigma^n X)$
(May, classifying spaces & fib
Barratt-Priddy (1978)
grp cplt. Hauschild (~~unpublished~~) (2001)

$H_*(C_n X, \mathbb{F}_p)$
$H_*(\Omega^n\Sigma^n X, \mathbb{F}_p)$
are functors of $H_*(X, \mathbb{F}_p)$.

$n=1$ James (1955)

$n<\infty$ Milgram (connected CW) 1966

$n=\infty$ Dyer-Lashof (unpublished)

Barratt (1970)

Quillen (unpublished)

Segal ~~(1968)~~ ~~homotopy~~ everything h-spaces. May, GILS

Ronke, Sanderson 1985

Caruso, Waner $\left(\begin{matrix}\text{compact } G \\ \mathbb{R}^\infty \subseteq V\end{matrix}\right)$

Fred cohen

Segal $X = S^0$ (1973)

Reading the theorem. $\mathcal{C}_n(j) \simeq \mathcal{F}_{\mathbb{R}^n}(j)$ ordered configuration space of j points

$F_{\mathbb{R}^n} := \coprod_{j \geq 0} \mathcal{F}_{\mathbb{R}^n}(j)/\Sigma_j \xrightarrow{\text{grp cplt}} \Omega^n\Sigma^n S^0 = \Omega^n S^n$ (Segal)

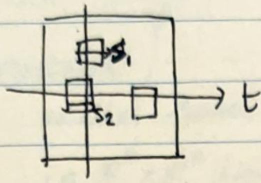
$n=1$ on interval, $\mathcal{F}_{\mathbb{R}^1}(j) \simeq *$ $\rightsquigarrow \mathbb{N} \to \Omega^1 S^1 \simeq \mathbb{Z}$. $n=2$. $n=\infty$.

$F_{\mathbb{R}^n}(X) \xrightarrow{\sim} \Omega F_{\mathbb{R}^{n-1}}(\Sigma X) \xrightarrow{\sim} \Omega^2 F_{\mathbb{R}^{n-2}}(\Sigma^2 X) \xrightarrow{\sim} \cdots \xrightarrow{\text{grp}} \Omega^n\Sigma^n X$.

$BF_{\mathbb{R}^n}(X) \simeq \cancel{\#} F_{\mathbb{R}^{n-1}}(\Sigma X)$

$\beta_n : C_n \to \Omega C_{n+1}\Sigma$

$\alpha_n : C_n \xrightarrow{\beta_n} \Omega C_{n+1}\Sigma \xrightarrow{\Omega\alpha_{n+1}\Sigma} \Omega^n\Sigma^n$



$c \in \mathcal{C}_n(j)$, $x_i \in X$, $t \in [0,1]$

$\beta_n[c, (x_i)](t) = [c_t, (x_i \wedge s_i)]$

$s_i \in [0,1]$, $c_t \in \mathcal{C}_{n+1}(-)$.

Consequence.

$B(C_n, C_n, X) \xrightarrow[\text{(grp cplt)}]{\sim} B(\Omega^n\Sigma^n, C_n, X) \xrightarrow{\sim} \Omega^n B(\Sigma^n, C_n, X)$

commuting mapping space with geometric realization

need $(\Sigma^n C_n^\bullet X)$ to be n-connective

<which is always true>

$$\begin{array}{ccccc} |\Omega_* X| & \longrightarrow & |P_* X| & \longrightarrow & |X| \\ \downarrow & & \downarrow & & \| \\ \Omega_*|X| & \longrightarrow & P|X| & \longrightarrow & |X| \end{array}$$

$*$ S^1

← quasifibration if each X_q is connected

Other operads.

Def 3.8. $\mathcal{C}, \mathcal{C}'$ operads $\rightsquigarrow (\mathcal{C} \times \mathcal{C}')(j) = \mathcal{C}(j) \times \mathcal{C}'(j)$

Def 3.9. $\mathcal{C}, \mathcal{C}'$ operads over $\mathcal{M}$ $\rightsquigarrow \mathcal{C} \triangledown \mathcal{C}'$

$$\begin{array}{ccc} \mathcal{C} \triangledown \mathcal{C}' & \longrightarrow & \mathcal{C}' \\ \downarrow & & \downarrow \varepsilon' \\ \mathcal{C} & \xrightarrow[\varepsilon]{} & \mathcal{M} \end{array}$$

fibered product of $\varepsilon, \varepsilon'$ in the category of operads.

Prop 3.10. (1) $\mathcal{C}$: A_∞ operad $\pi_0 \mathcal{C}(n) \cong \Sigma_n$ and components contractible.

$\mathcal{C}' \to \mathcal{M}$

Then $\pi_2 : \mathcal{C} \triangledown \mathcal{C}' \to \mathcal{C}'$ is an equivalence of operads.

(2) $\mathcal{C}$: E_∞ operad

Then $\pi_2 : \mathcal{C} \times \mathcal{C}' \to \mathcal{C}'$ is an . . . .

Can use π_2 to change operads

Cor 6.2 (1) $\mathcal{C}$: A_∞. $MX \xleftarrow{\varepsilon} CX \xleftarrow{\pi_1} (C \triangledown C_1)X \xrightarrow{\pi_2} C_1 X \xrightarrow{\alpha_1} \Omega\Sigma X$.

$X \in \mathcal{T}_{\geq 0}$ (2) $\mathcal{C}$: E_∞. $CX \leftarrow (C \times C_\infty)X \to C_\infty X \to \Omega^\infty \Sigma^\infty X$.

Geometric proof [Rourke–Sanderson]

Goal. X connected $\Rightarrow$ $C_nX \to \Omega^n\Sigma^n X$ is w.e.

i.e. $\underbrace{\pi_k C_n X}_{} \xrightarrow{\cong} \underbrace{\pi_{k+n}\Sigma^n X}_{}$

representative:

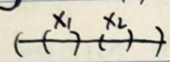
parallel framed manifold in $\mathbb{R}^k\times\mathbb{R}^n$ labelled in X

semi-parallel framed manifold in $\mathbb{R}^k\times\mathbb{R}^n$ labelled in X.

slogan: every semi-parallel mfld can be made parallel.

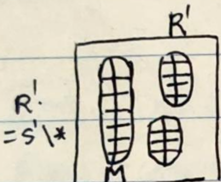
$n=1$ $\quad C_1X \to \Omega^1\Sigma^1X$ $\quad$ consider: $[Q, C_1X] \to [Q, \Omega\Sigma X]$

Q smooth manifold



$\in C_1X$

$\boxed{Map_*(Q, C_1X)}$



$\in Map_*(S^1, C_1X) = Q$

center of cubes

"$M \subseteq Q\times\mathbb{R}^1$ smooth with ∂

+ $M \to Q$ is local diffeomorphism / homeo

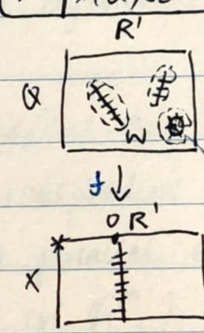
+ normal bundle ν_M framed canonically in the $\mathbb{R}^1$ direction

+ label $l: (M, \partial M) \to (X, *)$

or $l: (\nu_M, \partial\nu_M) \to (X\times\mathbb{R}^1, *)$

$\boxed{Map_*(Q, \Omega\Sigma X)}$

$Q = Map_*(S^1, \Omega\Sigma X) \cong Map_*(S^1\wedge S^1, \Sigma X)$



closed component, can be punctured by moving in the label space X to the base point.

$f^{-1}(\{0\}\times X\backslash *)$

"$W \subseteq Q\times\mathbb{R}^1$ smooth with ∂.

$f^{-1}(\Sigma X\backslash *)$

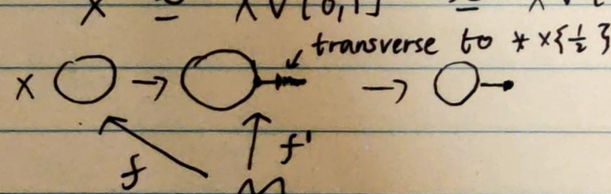
+ normal bundle ν_W framed

+ $l: (W, \partial W) \to (X, *)$

or $l: (\nu_W, \partial\nu_W) \to (X\times\mathbb{R}^1, *)$

Whisker trick: $f: \overset{\text{smooth}}{M} \to \overset{\text{space}}{X}$, can modify f up to homotopy such that $f^{-1}(X\backslash *)$ is a smooth manifold $\subseteq M$ with $\partial \mapsto *$.

$X \simeq X\vee[0,1] \simeq X\vee[0,\tfrac{1}{2}]$



Compression theorem (paper I, Thm 2.1)

$M^m \subseteq Q^q \times R$ embedded with a normal vector field and $q-m \geq 1$, Then the vector field can be made parallel in the given R direction by an isotopy of M and normal field in $Q \times \mathbb{R}$.

Addenda. (i) $C \subseteq Q$ compact. If M is already compressible in a neighborhood of $C \times R$, then the isotopy can be assume fixed on $C \times \mathbb{R}$.

(iii) $q-m=0$ ok with some more assumptions (vector field is "$\perp$" and "grounded")

$$\text{Compression} + \begin{cases} Q = S^k & \rightsquigarrow \pi_k(\alpha_1) \text{ surjective} \\ Q = S^k \times I & \rightsquigarrow \pi_k(\alpha_1) \text{ injective} \end{cases}$$

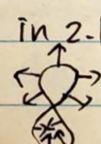
$\alpha_n;\ n > 1$

Multi-compression theorem (part I, Thm/Cor 4.5)

$M^m \subseteq Q^q \times \mathbb{R}^n$ embedded with n independent normal vector field, $q-m \geq 1$, Then M is isotopic to a parallel embedding (i.e. norm vector fields are parallel to coordinates in $\mathbb{R}^n$.)

Addenda. (ii) If every component of M has relative boundary then the dimension condition can be relaxed to $q-m \geq 0$.

(Remark. Addenda (iii) in 2.1 does not work for immersions ~~here~~, see)



gronded & parallel $\rightsquigarrow$ can not be made compressible as S^1 does not immerse in R^1.

GILS.

quasi-fibration, X connected.

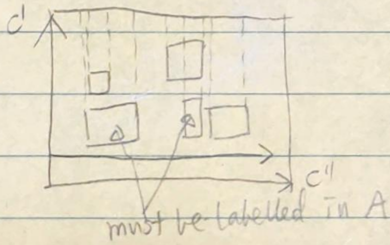
Thm 6.1.
$$\begin{tikzcd} C_n X \arrow[r, "c"] \arrow[d, "\alpha_n"'] & E_n X \arrow[r, "\pi_n"] \arrow[d, "\tilde{\alpha}_n"] & C_{n-1}\Sigma X \arrow[d, "\alpha_{n-1}"] \\ \Omega^n \Sigma^n X \arrow[r, "c"] & P\Omega^{n-1}\Sigma^n X \arrow[r, "p"] & \Omega^{n-1}\Sigma^n X \end{tikzcd}$$
$E_n X \simeq *$

Construction 6.6. $E_n X := E_n(CX, X) \to E_n(\Sigma X, *) \xrightarrow{\pi} C_{n-1}(\Sigma X)$

where for $A \subseteq X$,

- $E_n(X, A) \subseteq C_n(X)$

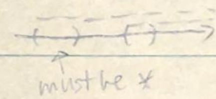
  $(\langle c_1, \cdots, c_j \rangle, x_1, \cdots, x_j)$ s.t. if $x_r \notin A$, then $(c_r'(0), 1) \times c_r''(I^{n-1})$ intersects $c_s(I^n)$ trivially for $s \neq r$.



- $\pi : E_n(A, *) \to C_{n-1}(A)$.

  $(\langle c_1, \cdots, c_j \rangle, x_1, \cdots, x_j) \in E_n(A,*)$

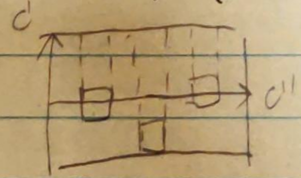
  if $n=1$,



  $\pi : E_1(A, *) \simeq C_1(1) \times X/\sim \xrightarrow{fgt} X$

  If $n>1$, take a representative class s.t. on x_i is $*$.

  $\pi(\ ) = (\langle c_1'', \cdots, c_j'' \rangle, x_1, \cdots, x_j)$ (forget the first direction)



- $(CX, X) \xrightarrow{\tilde{\beta}_n} (P\Omega^{n-1}\Sigma^{n-1}X, \Omega^n\Sigma^n X)$
- $E_n(P\Omega^{n-1}\Sigma^{n-1}X, \Omega^n\Sigma^n X) \xrightarrow{\tilde{\theta}_n} P\Omega^{n-1}\Sigma^n X$