Talk 7: Operad pairs, the Steiner and linear isometries operads.

Main reference: J P May, What precisely are E_∞ ring spaces and E_∞ ring spectra.

The main topic of this talk is operad pairs. Before we discuss that, let us have a quick review of the notion of operads.

1. Review of operads.

Let $\mathcal{U}$ denote the category of (compactly generated) unbased spaces. An operad $\mathcal{O}$ in $\mathcal{U}$ consists of spaces $\mathcal{O}(j) \in \mathcal{U}$ for $j \geq 0$, such that (a) $\mathcal{O}(0)$ is a point $*$,

(b) there is an element $\mathrm{id} \in \mathcal{O}(1)$ that corresponds to the identity operation,

(c) $\mathcal{O}(j)$ has a right action of the symmetric group Σ_j.

There are structure maps

$$\gamma: \mathcal{O}(k) \times \mathcal{O}(j_1) \times \cdots \quad \mathcal{O}(j_k) \to \mathcal{O}(j_1 + \cdots + j_k)$$

that are suitably equivariant, unital, and associative.

We say that $\mathcal{O}$ is an E_∞ operad if $\mathcal{O}(j)$ is contractible and Σ_j acts freely.

Let $X \in \mathcal{U}$, an action θ of $\mathcal{O}$ on X is given by suitably equivariant, unital, and associative action maps:

$$\theta : \mathcal{O}(j) \times X^j \to X$$

We can think of $\mathcal{O}(j)$ as parametrizing a j-fold product operation on X. Let $j=0$, we can think of $\theta : * \to X$ as giving X a basepoint.

2. Operad pairs

We can use operad pairs to encode two different operations on an object. We could gain some motivation from the classical algebraic definition of semirings.

<u>Def.</u> A semiring is a set R equipped with two binary operations $+$ and $\cdot$, called addition and multiplication, such that

- $(R, +)$ is a commutative monoid with identity element called 0 $((a+b)+c = a+(b+c) ; 0+a = a = a+0 ; a+b = b+a)$
- $(R, \cdot)$ is a monoid with identity element called 1

  $((a \cdot b) \cdot c = a \cdot (b \cdot c) ; 1 \cdot a = a = a \cdot 1)$

- Multiplication by the additive identity 0 annihilates R
  $(a \cdot 0 = 0 = 0 \cdot a)$
- Multiplication left and right distributes over addition
  $(a \cdot (b+c) = a \cdot b + a \cdot c \ ; \ (b+c) \cdot a = b \cdot a + c \cdot a)$

With this algebraic definition in mind, we can define operad pairs. Instead of giving you the precise definition, let me just tell you the intuition so that you can appreciate the simple idea behind it. The precise definition can be found in, for example [May, The construction of E_∞ ring spaces from bipermutative categories, Section 4]

"Def" Let $\mathcal{C}$, $\mathcal{G}$ be operads. Write $\mathcal{C}(0) = \{0\}$, $\mathcal{G}(0) = \{1\}$

An action of $\mathcal{G}$ on $\mathcal{C}$ consists of maps:

$$\lambda : \mathcal{G}(k) \times \mathcal{C}(j_1) \times \cdots \times \mathcal{C}(j_k) \to \mathcal{C}(j_1 \cdot j_2 \cdots j_k)$$

for $k \geq 0$ and $j_r \geq 0$ which satisfy certain distributivity, unity, equivariance, and nullity properties.

Here, we think of $\mathcal{C}$ as parametrizing addition and $\mathcal{G}$ as parametrizing multiplication.

The intuition for taking products of j's on the right hand side comes from the simple example

$$(x+y)(a+b+c) = xa + xb + xc + ya + yb + yc$$

<u>Def.</u> An operad pair $(\mathcal{C}, \mathcal{G})$ consists of operads $\mathcal{C}, \mathcal{G}$, together with an action of $\mathcal{G}$ on $\mathcal{C}$.

<u>Def</u> An action of $(\mathcal{C}, \mathcal{G})$ on X consists of an action θ of $\mathcal{C}$ on $(X, 0)$ and an action ξ of $\mathcal{G}$ on $(X, 1)$ for which the following diagram commutes.

$$\begin{tikzcd}
\mathcal{G}(k) \times \mathcal{C}(j_1) \times X^{j_1} \times \cdots \times \mathcal{C}(j_k) \times X^{j_k} \arrow[r, "\mathrm{id} \times \theta^k"] \arrow[d, "\xi"'] & \mathcal{G}(k) \times X^k \arrow[d, "\xi"] \\
\mathcal{C}(j_1 j_2 \cdots j_k) \times X^{j_1 j_2 \cdots j_k} \arrow[r, "\theta"] & X
\end{tikzcd}$$

where the ξ on the left is defined by

$$\xi(g; c_1, y_1, \cdots, c_k, y_k) = (\lambda(g, c_1, \dots, c_k); \prod_Q \xi(g; y_Q))$$

where $g \in \mathcal{G}(k)$, $c_r \in \mathcal{C}(j_r)$, $y_r = (x_{r,1}, \cdots, x_{r,j_r})$

Q is the lexicographically ordered set of sequences $Q = (q_1, \cdots, q_k)$ such that $1 \leq q_r \leq j_r$, and

$y_Q = (x_{1,q_1}, \cdots, x_{k,q_k})$.

The diagram encodes the left distributivity law.

Example: Recall operad $\mathcal{N}$ with $\mathcal{N}(j) = *$ for all j.

There is one and only one way $\mathcal{N}$ can act on itself.

An $(\mathcal{N}, \mathcal{N})$ space is precisely a commutative topological semi-ring.

Def We say $(\mathcal{C}, \mathcal{G})$ is an E_∞ operad pair if $\mathcal{C}$ and $\mathcal{G}$ are both E_∞ operads. An E_∞ ring space is a $(\mathcal{C}, \mathcal{G})$-space where $(\mathcal{C}, \mathcal{G})$ is some E_∞ operad pair.

3. The canonical E_∞ operad pair $(\mathcal{C}, \mathcal{L})$

The canonical multiplicative operad is the linear isometries operad $\mathcal{L}$. We now recall its definition.

Let $\mathcal{J}$ denote the category of finite dimensional real inner product spaces and linear isometric isomorphisms. Let $\mathcal{J}_c$ denote the category of finite or countably infinite dimensional real inner product spaces and linear isometries. For the latter, we topologize inner product spaces as the colimits of their finite dimensional subspaces.

Let $\mathcal{U} = \mathbb{R}^\infty$ with its standard inner product. Define $\mathcal{L}(j) = \mathcal{I}_c(\mathcal{U}^j, \mathcal{U})$, where $\mathcal{U}^j$ is the sum of j copies of $\mathcal{U}$, with $\mathcal{U}^0 = \{0\}$. The element $id \in \mathcal{L}(1)$ is the identity isometry, Σ_j acts on $\mathcal{L}(j)$ by permuting the inputs in $\mathcal{U}^j$, and the structure maps γ are defined by

$$\gamma(g; f_1, \cdots, f_j) = g \circ (f_1 \oplus \cdots \oplus f_j)$$

Notice that $\mathcal{L}$ is a suboperad of the endomorphism operad of $\mathcal{U}$. Clearly the Σ_j action on $\mathcal{L}(j)$ is free. One can also show that $\mathcal{L}(j)$ is contractible [May, E_∞ ring spaces and E_∞ ring spectra, Lemma 1.3]

<u>Prop</u> $\mathcal{L}$ is an E_∞ operad.

The canonical additive E_∞ operad is similar to the little cubes operads $\mathcal{C}_n$ and the little discs operads $\mathcal{D}_V$. It combines the good properties of both.

Let's first recall the general definition of embeddings operad.

Let X be an open subspace of a finite dimensional inner product space V. Define the embeddings operad Emb_X as follows. Let $Emb_X(j)$ be the space of j-tuples of embeddings with disjoint images. The element $id \in Emb_X(1)$ is the identity embedding. Σ_j acts on $Emb_X(j)$ by permuting embeddings. The structure maps

$$\gamma : Emb_X(k) \times Emb_X(j_1) \times \cdots \times Emb_X(j_k) \to Emb_X(j_1 + \cdots + j_k)$$

are defined by composition: for $g = (g_1, \cdots, g_k) \in Emb_X(k)$ and $f_r = (f_{r,1}, \cdots, f_{r,j_r}) \in Emb_X(j_r)$, $1 \leq r \leq k$, the rth block of j_r embeddings in $\gamma(g; f_1, \cdots, f_k)$ is given by $g_r \circ f_{r,s}$, $1 \leq s \leq j_r$.

Taking $X = (0,1)^n \subset \mathbb{R}^n$, we obtain a suboperad $\mathcal{C}_n$ of Emb_X by restricting to the little n-cubes, namely those embeddings

$f : X \to X$ such that $f = l_1 \times \cdots \times l_n$, where $l_i(t) = a_i t + b_i$, $a_i > 0$, $b_i \geq 0$.

For general V, let X be the open unit disk $D(V) \subset V$. We obtain a suboperad $\mathcal{D}_V$ of Emb_V by restricting to the little V-disks namely those embeddings $f: D(V) \to D(V)$ such that $f = av + b$, $a > 0$, $b \in D(V)$

Let $F(X, j)$ denote the configuration space of j-tuples of distinct elements of X, with its permutation action by Σ_j. By restricting the little n-cubes or little V-disks to their values at the center point, we obtain Σ_j-equivariant deformation retractions.

$$\mathcal{C}_n(j) \xrightarrow{ev_0} F((0,1)^n, j) \cong F(\mathbb{R}^n, j)$$

$$\mathcal{D}_V(j) \xrightarrow{ev_0} F(D(V), j) \cong F(V, j)$$

This gives control over homotopy types.

Prop. $F(\mathbb{R}^n, j)$ is $(n-2)$-connected. Let $\mathbb{R}^\infty = \operatorname{colim}_n \mathbb{R}^n$.

then $F(\mathbb{R}^\infty, j) = \operatorname{colim}_n F(\mathbb{R}^n, j)$ is Σ_j-free and contractible.

[See, for example, May: The geometry of iterated loop spaces Section 4].

The little n-cube operads

☺. If f is a little n-cube, then $f \times \mathrm{id}$ is a little $(n+1)$-cube. This induces a suspension map of operads $\mathcal{C}_n \to \mathcal{C}_{n+1}$.. Taking colimits over n gives the infinite little cubes operad $\mathcal{C}_\infty$, and it is an E_∞ operad.

☹: little n-cubes are too square to define $O(n)$ actions.

The little V-disk operad $\mathcal{D}_V$

☺: Let f be a little V-disk, $g \in O(V)$, gfg^{-1} is also a little V-disk.

☹: For $W = V \oplus V^{\perp}$, little V-disks f are too round for $f \times \mathrm{id}$ to be a little W-disk. We can send a little V-disk $v \mapsto av+b$ to the little W-disk $w \mapsto aw+b$. but that is not compatible with the decomposition $S^W \cong S^V \wedge S^{V^{\perp}}$ used to identify $\Omega^W Y$ with $\Omega^{V^{\perp}} \Omega^V Y$. Here, we let S^V denote the one point compactification of V and let $\Omega^V Y = F(S^V, Y)$ denote the V-fold loop space of Y.

Steiner operads $\mathcal{K}_V$ combine all of the good properties of $\mathcal{C}_n$ and $\mathcal{D}_V$. These operads are defined in terms of paths of embeddings rather than just embeddings.

Let $R_V \subset \mathrm{Emb}_V(1)$ be the subspace of distance reducing embeddings $f: V \to V$, i.e. $|f(v) - f(w)| \leq |v-w|$, $\forall v, w \in V$. A Steiner path is a map $h: I \to R_V$ such that $h(1) = id$. Let P_V be the space of Steiner paths. Define $\pi: P_V \to R_V$ by evaluating at 0, $\pi(h) = h(0)$. Define $\mathcal{K}_V(j)$ to be the space of j-tuples $(h_1, \cdots, h_j)$ of Steiner paths such that the $\pi(h_r)$ have disjoint images. $id \in \mathcal{K}_V(1)$ is the constant path at the identity embedding, Σ_j acts on $\mathcal{K}_V(j)$ by permutations, the structure maps γ are defined by composition.

Again, it is clear that the Σ_j action is free. Moreover, Steiner proved the composite maps $\mathcal{K}_V(j) \xrightarrow{\pi} \mathrm{Emb}_V(j) \xrightarrow{ev_0} F(V, j)$ are Σ_j-equivariant deformation retractions.

Finally, we define the canonical additive E_∞ operad, denoted $\mathcal{C}$, to be the Steiner operad $K_{\mathcal{U}} = K_{\mathbb{R}^\infty} = \operatorname{colim}_V K_V$ with V running through the finite dimensional subspaces of $\mathbb{R}^\infty$. Then, analogous to the little cubes operad $\mathcal{C}_\infty$, $\mathcal{C}$ is also an E_∞ operad.

<u>Prop.</u> $(\mathcal{C}, \mathcal{L})$ is an E_∞ operad pair.

<u>Intuition</u>: we have maps

$$\lambda : \mathcal{J}(V_1 \oplus \cdots \oplus V_k, W) \times \mathrm{Emb}_{V_1}(j_1) \times \cdots \times \mathrm{Emb}_{V_k}(j_k) \to \mathrm{Emb}_W(j_1 \cdots j_k)$$

defined as follows: let $g : V_1 \oplus \cdots \oplus V_k \to W$ be a linear isometric isomorphism. Let $f_r = (f_{r,1}, \cdots, f_{r,j_r}) \in \mathrm{Emb}_{V_r}(j_r)$, $1 \le r \le k$. Consider the set of sequences $Q = (q_1, \cdots, q_k)$, $1 \le q_k \le j_r$, ordered lexicographically. Identifying direct sums with direct products, the Q-th embedding of $\lambda(g; f_1, \cdots, f_k)$ is the composite $g f_Q g^{-1}$ where $f_Q = f_{1,q_1} \times \cdots \times f_{k,q_k}$.