Meng Guo 2023-08-15

Taken by: Jialiang Lin

(C, G)

$\lambda: G(k) \times C(j_1) \times \cdots \times C(j_k) \to C(j_1 j_2 \cdots j_k)$

A (C,G)-space is (X, θ, ξ)

s.t. (X, θ) is C-space and (X, ξ) is G-space

with a map

$$\xi_k: G(k) \times C(j_1) \times X^{j_1} \times \cdots \times C(j_k) \times X^{j_k} \to C(j_1 \cdots j_k) \times X^{j_1 \cdots j_k}$$

$$\xi_k(g, c_1, y_1, \cdots, c_k, y_k) = (\lambda(g c_1 c_2 \cdots c_k), \prod_Q \xi(g, y_Q))$$

such that

$$\begin{tikzcd}
G(k) \times C(j_1) \times X^{j_1} \times \cdots \times C(j_k) \times X^{j_k} \arrow[r, "id \times \theta^k"] \arrow[d, "\xi"'] \arrow[dr, phantom, "\circlearrowright"] & G_k \times X^k \arrow[d, "\xi"] \\
C(j_1 \cdots j_k) \times X^{j_1 \cdots j_k} \arrow[r, "\theta"'] & X
\end{tikzcd}$$

E.g.

$k=2, \ j_1=3, \ j_2=2$

$$(g, c_1, (x_1, x_2, x_3), c_2(x_1', x_2')) \xmapsto{\xi_2} (-, \xi g(x_1, x_1'), \xi g(x_1, x_2'), \\ \xi g(x_2, x_1'), \xi g(x_2, x_2'), \\ \xi g(x_3, x_1'), \xi g(x_3, x_2'))$$

$$(x_1 + x_2 + x_3)(x_1' + x_2') = x_1 x_1' + x_1 x_2' + \cdots + x_3 x_1' + x_3 x_2'$$

$C \quad C(0) \times X^0 \to X \ni 0$

$G \quad G(0) \times X^0 \to X \ni 1$

RMk/Def (C, G) E_∞ pair

(C, G) : E_∞-ring space.

(C,G)-space is C-algebra in cat of G_+-space

$G_+(j) = G(j)_+$

Prop.

C is a monad when restricted to G_+-space $\subset Top_*$

So (C,G)-space is C-algebra in G_+-spaces

Recall

$CX = \coprod C(j) \times X^j / \sim$

Pf

$$\begin{array}{c} G(k) \times CX \times \cdots \times CX \\ \downarrow \\ C(X) \end{array}$$

check

$$\left.\begin{array}{l} \mu : CCX \to CX \\ \eta : X \to CX \end{array}\right\} \text{ if } X \text{ is a } G_+\text{-space}$$

$$X \underset{\dashrightarrow}{\overset{w.e.}{\longleftarrow}} B(C,C,X) \xrightarrow{\alpha} B(\Omega^\infty\Sigma^\infty, C, X) \to \Omega^\infty B(\Sigma^\infty, C, X)$$

Top$_*$

$$\begin{tikzcd}
C\text{-space} \arrow[d, "\begin{array}{l}\text{additive}\\ \infty\text{-loop}\\ \text{machine}\end{array}"'] \\
\infty\text{-loop space} \arrow[d, squiggly] \\
\text{Spectrum}
\end{tikzcd}$$

$(\Sigma^\infty, \Omega^\infty)$ Spectrum $(\Sigma^\infty X, Y) \simeq$ Space$(X, \Omega^\infty Y)$

Ω^∞(Spectrum) $=$ E_∞-space

G_+-space

$$\begin{tikzcd}
(C,G)\text{-space} \arrow[d, "\begin{array}{l}\text{multiplicative}\\ \infty\text{-loop}\\ \text{machine}\end{array}"] \\
E^\infty\text{-ring spectrum}
\end{tikzcd}$$

Ω^∞(E_∞-ring Spectrum) $=$ E_∞-ring space

$$\begin{tikzcd}[column sep=small]
\text{Spectrum}(\Sigma^\infty X, Y) \arrow[d, phantom, "\overset{?}{\shortparallel}"] & \simeq & \text{Top}_*(X, \Omega^\infty Y) \arrow[d, phantom, "\cup"] \\
 & \simeq & G_+\text{-space} \\
G_+\text{-algebra in Spectrum} & &
\end{tikzcd}$$

Now, change G to L!

$$\begin{tikzcd}[column sep=small]
\text{Spectrum}(\Sigma^\infty X, Y) & \simeq & \text{Top}_*(X, \Omega^\infty Y) \arrow[d, phantom, "\cup"] \\
L_+\text{-algebra in Spectrum} & \simeq & L_+\text{-space}
\end{tikzcd}$$

$\mathcal{L}$-Spectrum / E_∞-ring Spectrum

Spectrum R, $R \wedge R \to R$

$\mathcal{L}(j) \times R^{\wedge j} \to R$

associative, unit, equavariant.

Prop.

$\Omega^\infty \Sigma^\infty$ is monad in $\mathcal{L}_+$-space.

E_V $V \in \mathcal{U} = \mathbb{R}^\infty$

$\sigma : E_V \to \Omega^{W-V} E_W$ $V \subset W$

$\Omega^V X$ Σ^V

$\Omega^V \Sigma^V$

$A(V, W)$

$A(V,W) \times V \to A(V,W) \times W$, $V \subset W$

$(f, v) \mapsto (f, fv)$

$E(V,W) \triangleq$ complementary of subbundle

$T(V,W) =$ Thom space.

$(\mathcal{L}_+(j) \ltimes E^{\wedge j})_W = T(V,W) \wedge E_{V_1} \wedge \cdots \wedge E_{V_j}$