$H_*(CX)$ and $H_*(\Omega^n\Sigma^n X)$

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For a prime p, write $H_*X = H_*(X;\mathbb{F}_p)$ and $H^*(X;\mathbb{F}_p)$. We assume $p = 2$ but everything in this notes has odd prime versions that you can find in *Homology of iterated loop spaces* and *A general algebraic approach to Steenrod operations*.

1 Steenrod operations

For any space X, we have Steenrod operations

$$Sq^s : H^*(X) \to H^{*+s}(X)$$

The Steenrod operations satisfy the Adem relations:

$$Sq^i Sq^j = \sum_k (i-2k, j-k+i-1) Sq^{i+j-k} Sq^k$$

where $i < 2j$ and we have $Sq^1 = 1$. Here $(m,n) = \binom{m+n}{n}$ and is trivial if $m < 0$ or $n < 0$.

The Steenrod algebra $\mathcal{A}$ is generated by Sq^s with Adem relations. Hence cohomology groups $H^*(X)$ are modules over $\mathcal{A}$, and homology groups $H_*(X)$ are (left) modules over the opposite algebra $\mathcal{A}^{op}$.

2 Homology operations

Homology operations are also called Dyer-Lashof operations.

Theorem 2.1. *Let $\mathscr{C}$ be an E_∞ operad and X a $\mathscr{C}$-space. Then we have homomorphisms*

$$Q^s : H_*(X) \to H_{*+s}X$$

such that

1. *Q^s are natural with respect to $\mathscr{C}$-spaces.*
2. *$Q^s x = 0$ if $s < |x|$.*
3. *$Q^s x = x^p$ if $s = |x|$.*
4. *$Q^s[e] = 0$ if $s > 0$ and $[e] \in H_0(X)$ is the identity element.*
5. *$Q^s(x \otimes y) = \sum_{i+j=s} Q^i \otimes Q^j$ where $x \otimes y \in H_*(X \times y)$.*
   $Q^s(xy) = \sum_{i+j=s} Q^i x Q^j y$ where $x, y \in H_*(X)$.
   $\psi(Q^s x) = \sum \sum_{i+j=s} Q^i x' \otimes Q^j x''$ where ψ is the coalgebra structure map of $H_*(X)$ and $\psi(x) = \sum x' \otimes x''$.

6. The Adem relations hold. If $2r > s$, we have

$$Q^r Q^s = \sum_i (-1)^{r+i}(pi - r, r - (p-1)s - i - 1) Q^{r+s-i} Q^i$$

7. The Nishida relations hold.

$$P_*^r Q^s = \sum_i (-1)^{r+i}(r - pi, s(p-1) - pr + pi) Q^{s-r+i} P_*^i$$

The homology operations can be defined by the structure map passing to the homology:

$$H_*(\mathscr{C}(p) \otimes X^p) \to H_* X$$

$$e_i \otimes x^p \mapsto Q^* x$$

The Adem relations can be proved by maps in homology induced by the commutative diagram

$$\begin{array}{ccccc}
\mathscr{C}(p) \times \mathscr{C}(p)^p \times X^{p^2} & \longrightarrow & \mathscr{C}(p^2) \times X^{p^2} & & \\
 & & & \searrow & \\
\big\downarrow & & & & X \\
 & & & \nearrow & \\
\mathscr{C}(p) \times (\mathscr{C}(p) \times X^p)^p & \longrightarrow & \mathscr{C}(p) \times X^p & &
\end{array}$$

Basically you can start from an element in $H_*(\mathscr{C}(p) \times \mathscr{C}(p)^p \times X^{p^2})$ and get two elements in $H_* X$ which should be equal.

You can find proofs of other items from the two references.

We define R to be the algebra generated by Q^s with the Adem relations.

If a (graded) R-module M satisfy the condition 2 in Theorem 1, we say that it is an allowable R-module. If M enjoys all the structures and properties we see in $H_*(X)$ in Theorem 1, we call it an allowable AR-Hopf algebra.

3 $H_*(CX)$

For any space X, $H_* X$ is a cocommutative component unstable A-coalgebra. Since CX is an E_∞-space, we know that $H_*(CX)$ is an allowable AR-Hopf algebra.

There is a forgetful functor from the category of allowable AR-Hopf algebras to the category of cocommutative component unstable A-coalgebras. It is not hard to construct the left adjoint which is the free functor from the category of cocommutative component unstable A-coalgebras to the category of allowable AR-Hopf algebras. We denote this free functor by WE.

Theorem 3.1. *There is a natural isomorphism $H_*(CX) \cong WEH_*(X)$.*

When Y is a group like E_∞ algebra, $H_* Y$ is an allowable AR-Hopf algebra with conjugation χ. For $g \in H_0 Y$ we have $\chi(g) = g^{-1}$.

Theorem 3.2. *There is a natural isomorphism $H_*(QX) \cong GWEH_*(X)$ where G is the free functor from the category of allowable AR-Hopf algebras to the category of allowable AR-Hopf algebras with conjugation.*

Remark 3.3. There are similar theorems about $H_*(C_n X)$ and $H_*(\Omega^n \Sigma^n X)$, who are both functors of $H_* X$.

Remark 3.4. If $(\mathscr{C}, \mathscr{G})$ is an operad pair and $\mathscr{G}$ acts on X, then CX is an E_∞ ring space and $H_*(CX)$ is equipped with two sets of Dyer-Lashof operations $\{Q^s\}$, $\{\tilde{Q}^s\}$. These operations interact with each other via formulas such as the mix Adem relations. We can also state theorems about $H_*(CX)$ in terms of H_*X in this situation.

Remark 3.5. If X is an E_∞ ring space, then H_*X is a Hopf ring which is equipped with a coalgbera structure and two algebra structure. In other words, we have an additive multiplication $\#$ and a multiplicative multiplication $\circ$. They satisfy the distributivity law

$$(r\#s) \circ t = \sum (r \circ t')\#(s \circ t'')$$

where $\psi(t) = \sum t' \otimes t''$.