Cohomology Theories and Naive Spectra

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1 Introduction

We will review Eilenberg-Steenrod axioms for cohomology. We construct Eilenberg-MacLanes spaces, which represent cohomology theories. This leads to naive $\Omega$-spectra and Brown representability.

2 Axiomatic cohomology

In this section we review Eilenberg-Steenrod axioms for cohomology theories.

A cohomology theory $E$ consists of $\mathbb{Z}$-graded contravariant functors $E^n$, from the category of pairs of CW complexes to the category of abelian groups, and natural transformations $\delta : E^n(A) := E^n(A, \emptyset) \Rightarrow E^{n+1}(X, A)$, such that:

- **Exactness.** The following sequence is exact:
  $$\cdots \to E^n(X, A) \to E^n(X) \to E^n(A) \to E^{n+1}(X, A) \to \cdots$$

- **Homotopy.** If $f : (X, A) \to (Y, B)$ is a homotopy equivalence, then
  $$f^* : E^n(Y, B) \xrightarrow{\cong} E^n(X, A).$$

- **Excision.** If $X$ is the union of subcomplexes $A$ and $B$, then the inclusion $(A, A \cap B) \to (X, B)$ induces an isomorphism
  $$E^n(X, B) \xrightarrow{\cong} E^n(A, A \cap B).$$

- **Additivity.** If $(X, A) = \bigsqcup_i (X_i, A_i)$, then
  $$E^n(X, A) \xrightarrow{\cong} \prod_i E^n(X_i, A_i).$$

Conventionally, if $E^n(\ast) = 0$ for $n \neq 0$, we call $E$ an ordinary cohomology theory.

There is a based variant: a reduced cohomology theory consists of $\mathbb{Z}$-graded functors $\tilde{E}^n$, from the category of based CW complexes to the category of abelian groups, and natural isomorphisms

$$\delta : \tilde{E}^n(X) \xrightarrow{\cong} \tilde{E}^{n+1}(\Sigma X),$$

such that:
• **Exactness.** If \( A \) is a subcomplex of \( X \) then the following sequence is exact:
\[ \tilde{E}^n(X/A) \to \tilde{E}^n(X) \to \tilde{E}^n(A). \]

• **Homotopy.** If \( f \simeq g : X \to Y \) are based homotopic, then \( f^* = g^* \).

• **Wedge.** If \( X = \bigvee_i X_i \), then
\[ \tilde{E}^n(X) \cong \prod_i \tilde{E}^n(X_i). \]

In this context the dimension axiom reads as \( \tilde{E}^n(S^0) = 0 \) for \( n \neq 0 \).

The relation between reduced and unreduced cohomology is the following:
\[ \hat{E}^*(X) = E^*(X, \ast), \quad E^*(X) = E^*(X_\ast), \quad E^*(X, A) = \hat{E}^*(X / A). \]

**Example.** Cellular/singular cohomology theory \( HG \).

**Example.** Ordinary cohomology of \( S^n \).

**Cup product and homology.**

3. **Eilenberg-MacLane spaces**

Given \( n > 0 \) and a discrete group \( G \), the **Eilenberg-MacLane space** \( K(G, n) \), is characterized by the following property: \( \pi_n K(G, n) = G \), while \( \pi_k K(G, n) = 0 \) for \( k \neq n \). Of course if \( n > 1 \) we require that \( G \) is abelian.

One way to construct Eilenberg-MacLanes spaces is by attaching cells. Say \( n \geq 1 \). Present \( G \) with generators and relations:
\[ G = \langle g_1, \ldots, g_\alpha / r_1, \ldots, r_\beta \rangle. \]

The homotopy group \( \pi_n(\bigvee_i S^{n_i}) \) is free abelian with \( \alpha \) generators. Each relation \( r_i \) is represented by a based map \( S^{n_i} \to \bigvee_i S^{n_i} \). One could attach a \((n + 1)\)-cell via this attaching map to realize the relation \( r_i \). The result is a space \( X \) with trivial homotopy groups \( \pi_i(X) \) for \( i < n \) and \( \pi_n(X) = G \).

The same method could be used to kill all higher homotopy groups. Starting with \( \pi_{n+1} \), we attach \((n + 2)\)-cells via attaching maps \( S^{n+1} \to X \) that generate \( \pi_{n+1} \), and this won’t affect lower homotopy groups. This finishes the construction.

Eilenberg-MacLane spaces are unique up to weak homotopy equivalence, some examples are
\[ K(\mathbb{Z}, 1) \simeq S^1, \quad K(\mathbb{Z}/2, 1) \simeq \mathbb{R}P^\infty, \quad K(\mathbb{Z}, 2) \simeq \mathbb{C}P^\infty. \]

Eilenberg-MacLane spaces represent cohomology theories. Recall that \([X, Y]\) denotes the set of based homotopy classes of maps between \( X \) and \( Y \), and \( \pi_0 F(X, Y) = [X, Y] \). The construction above reveals that Eilenberg-MacLane spaces are naturally based.

**Theorem.** For CW complexes \( X \), abelian groups \( G \), and integers \( n \geq 0 \), there are natural isomorphisms
\[ \hat{H}^n(X; G) \cong [X, K(G, n)]. \]

It is not hard to prove that for any based space \( Z \), the functor \([- , Z] \) from based CW complexes to pointed sets satisfies **Homotopy, Exactness** and **Wedge** conditions given in the Eilenberg-Steenrod
axioms for reduced cohomology theory. For the functor to take value in Abelian groups, we have to impose more structures on $Z$, for example if $Z$ is a double loop space. Milnor proved that the loop space of a CW complex has the homotopy type of a CW complex. Hence we have a homotopy equivalence

$$\tilde{\sigma}_n : K(G, n) \to \Omega K(G, n + 1).$$

By iterating, Eilenberg-MacLane spaces are infinity loop spaces.

An $\Omega$-spectrum is a sequence of based spaces $E_n$, $n \geq 0$, and based weak homotopy equivalences $\tilde{\sigma} : E_n \to \Omega E_{n+1}$. For an abelian group $G$, the Eilenberg-MacLane spectrum is $\{K(G, n), \tilde{\sigma}_n\}$.

**Proposition.** Let $E = \{E_n\}$ be an $\Omega$-spectrum. Define

$$E^n(X) = \begin{cases} [X, E_n] & \text{if } n \geq 0 \\ [X, \Omega^{-n}E_0] & \text{if } n < 0. \end{cases}$$

(3.1)

Then the functors $E^n$ define a reduced cohomology theory on based CW complexes. We only need to verify the suspension isomorphism, which is induced by $\tilde{\sigma}$:

$$E^n(X) = [X, E_n] \to [X, \Omega E_{n+1}] \cong [\Sigma X, E_{n+1}] = \tilde{E}^{n+1}(\Sigma X).$$

Now we have proved the theorem.

Cohomology could as well be generated to the $\infty$-categorical setting. The idea is that: given an $\infty$-category $\mathcal{C}$. For two objects $X, A$ of $\mathcal{C}$, the degree 0 cohomology of $X$ with coefficients in $A$, is the set of connected components of the hom space $\mathcal{C}(X, A)$.

We shall see Eilenberg-MacLane spaces also produce ordinary homology theories. By adjunction $[\Sigma X, Y] \cong [X, \Omega Y]$, $\tilde{e}_n : K(G, n) \to \Omega K(G, n+1)$ corresponds to map

$$\sigma_n : \Sigma K(G, n) \to K(G, n + 1).$$

We may smash with a based CW complex $X$ to obtain

$$\pi_{n+k}(X \wedge K(G, n)) \xrightarrow{\Sigma} \pi_{n+k+1}(X \wedge \Sigma K(G, n)) \xrightarrow{(\Id \wedge \sigma_n)_*} \pi_{n+k+1}(X \wedge K(G, n + 1)).$$

**Theorem.** For based CW complexes $X$, abelian groups $G$, and integers $n \geq 0$, there are natural isomorphisms

$$\tilde{H}_k(X, G) \cong \lim_{\text{colim}_n} \pi_{n+k}(X \wedge K(G, n)).$$

A spectrum is a sequence of based spaces $E_n$, $n \geq 0$, and based maps $\varphi_n : \Sigma E_n \to E_{n+1}$. Given nice conditions, one expect similar results. But we won’t go into details here. You will see an example at the beginning of next talk.

Now we build the Eilenberg-MacLane spaces into the construction of Postnikov towers which can be expressed as tower of fibrations with Eilenberg-MacLane spaces as fibers. We say a topological space is $n$-truncated if the homotopy groups of $X$ vanish in dimensions larger than $n$. Recall that the Postnikov tower of path-connected $X$, is a sequence of spaces

$$X \to \ldots \to X_n \xrightarrow{p_n} X_{n-1} \ldots \to X_1 \xrightarrow{p_1} X_0$$

such that
(1) \( \pi_i(X_n) \cong \pi_i(X) \) for \( i \leq n \).
(2) \( X_n \) is \( n \)-truncated, i.e., \( \pi_i(X_n) = 0 \) for \( i > n \).

We could construct a Postnikov tower by attaching cells when \( X \) is a CW complex. The Postnikov tower, if it exists, is unique up to homotopy.

Furthermore, one could successively replace each map \( p_n \) by a fibration: given a map \( f : X \to Y \), define the path space \( Nf = X \times_f Y^I \). \( Nf \) consists of pairs \( (x, \gamma) \) such that \( f(x) = \gamma(0) \). Now \( f \) could be decomposed as
\[
X \overset{v}{\to} Nf \overset{\rho}{\to} Y,
\]
where \( v(x) = (x, \gamma_f(x)) \) and \( \rho(x, \gamma) = \gamma(1) \). It is not hard to check that \( Nf \) deformation retracts to \( X \) and \( p \) is a fibration.

By examining the homotopy long exact sequence, the new map \( p'_n \) is a fibration with fiber \( K(\pi_n(X), n) \).

One recovers the space \( X \) by taking the homotopy limit of the tower. This kind of tower resolution construction is both theoretically and computationally important.

## 4 Brown Representability

On the other hand, the representability of ordinary cohomology is a consequence of a general result called the Brown representability theorem.

Recall that if \( \mathcal{C} \) is a category and \( F : \mathcal{C}^{\text{op}} \to \text{Set} \) is said to be representable if there exists \( X \in \mathcal{C} \) and an isomorphism \( F \cong \text{Hom}_{\mathcal{C}}(-, X) \).

There is a notion of presentable categories, as well as a notion of presentable \( \infty \)-categories.

**Proposition.** Let \( \mathcal{C} \) be a presentable category, and \( F : \mathcal{C}^{\text{op}} \to \text{Set} \) be a functor. \( F \) is representable if and only if \( F \) preserves limits.

**Proposition.** Let \( \mathcal{C} \) be a presentable \( \infty \)-category, and \( F : \mathcal{C}^{\text{op}} \to S \) be a functor. \( F \) is representable if and only if \( F \) preserves small limits.

There are also nice criteria (Adjoint Functor Theorem) to determine whether a functor between presentable (\( \infty \))-categories has left/right adjoints. As an example, the \( n \)-truncation functor is the left adjoint of the inclusion of \( \infty \)-category of \( n \)-truncated spaces into \( S \).

A contravariant functor from the homotopy category of based connected CW complexes to the category of pointed sets is called a Brown functor if it satisfies the following conditions:

1. it takes coproducts to products,
2. it takes weak pushouts to weak pullbacks.

**Theorem.** (Brown representability) Brown functors are representable. Every reduced cohomology theory on the category of based CW complexes is represented by an \( \Omega \)-spectrum.

## References