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# LECTURE 3-1: SYMMETRIC (BI)MONOIDAL CATEGORIES, (BI)PERMUTATIVE CATEGORIES AND THE STRICTIFICATION

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## 1. Introduction

Monoidal categories are categories with an addition and a unit satisfying associativity and unitality. Monoidal categories appearing in nature usually don't have strict associativity or unitality, but we can strictify to get an equivalent strict monoidal category.

When symmetry are involved, we can define braided monoidal categories and symmetric monoidal categories. Symmetric strict monoidal categories are also called permutative categories.

Permutative categories are inputs for the categorical infinite loop space machine. If you take the nerve of a permutative category and then realize, you get its classifying space, and the addition induces an addition on the classifying space which turns out to be $E_\infty$. We therefore obtain a categorical infinite loop space machine:

$$\text{Perm Cat} \xrightarrow{\text{classifying space}} E_\infty\text{–space} \xrightarrow{\text{infinite loop space machine}} \text{spectra.}$$

This story can also be enhanced to a multiplicative one.

We have a notion of bimonoidal categories, where the categories are equipped with two monoidal structures, one additive and one multiplicative. The definition also requires these two operations to have distributivity and annihilation, under certain coherence conditions. We also have a strictification theorem in the bimonoidal case, from strong symmetric bimonoidal categories to bipermutative categories.

Bipermutative categories are the input for the categorical multiplicative infinite loop space machine. One expect to have

$$\text{Biperm Cat} \xrightarrow{\text{classifying space}} E_\infty\text{-ring space} \xrightarrow{\text{machine}} E_\infty\text{-ring spectra.}$$

However, the first step is more complicated to show than the non-multiplicative case. It requires the notion of the categorical of operators, and will be introduced in the next lecture.

The purpose of this talk is to introduce the input of the categorical infinite loop space machine, i.e. symmetric (bi)monoidal categories and the (bi)permutative categories. We will also show the $E_\infty$-ness of the classifying space of a permutative category. We will also prove the strictfication theorems.

## 2. Monoidal category

A monoidal category is a category equipped with a "tensor product" satisfying associativity and unital properties.

**Definition 2.1** (Monoidal category)**.** A *monoidal category* is a category $\mathbf{C}$ equipped with the following monoidal structure satisfying certain coherence conditions.
(1) tesor product: a bifunctor

$$\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$$

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(2) a unit object: $I \in \mathbf{C}$,
(3) associator: natural isomorphism $\alpha$ with components

$$\alpha_{A,B,C} : A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C.$$

(4) left and right unitors: natural isomorphisms $\lambda$ and $\rho$ with components

$$\lambda_A : I \otimes A \simeq A,$$

$$\rho_A : A \otimes I \simeq A.$$

The coherence conditions for these natural transformations are the following: for all objects, the following diagram commutes.
(1) the pentagon diagram (all arrows are isomorphisms given by the associator):

$$\begin{array}{ccccc} & & A \otimes (B \otimes (C \otimes D)) & & \\ & \swarrow & & \searrow & \\ A \otimes ((B \otimes C) \otimes D) & & & & (A \otimes B) \otimes (C \otimes D) \\ \downarrow & & & & \downarrow \\ (A \otimes (B \otimes C)) \otimes D & & \longrightarrow & & ((A \otimes B) \otimes C) \otimes D \end{array},$$

(2) the triangle diagram

$$\begin{array}{ccc} A \otimes (I \otimes B) & \xrightarrow{\alpha_{A,I,B}} & (A \otimes I) \otimes B \\ & {\scriptstyle \rho_A \otimes \mathrm{id}_B} \searrow \quad \nearrow {\scriptstyle \mathrm{id}_A \otimes \lambda_B} & \\ & A \otimes B & \end{array}.$$

We sometimes write $(C, \otimes, I)$ to denote a monoidal category with tensor $\otimes$ and unite $I$.

There are some variants to such a structure.

**Definition 2.2.**

(1) A monoidal category with a commutativity constraint $\gamma$ is called a *braided monoidal category*. Here $\gamma$ is a natural isomorphism with components

$$\gamma_{A,B} : A \otimes B \to B \otimes A$$

and satisfies compatibility conditions with respect to $\alpha$ : the following diagram commutes (permuting $A$ across $B \otimes C$ can be done in two steps or at once).

$$\begin{array}{ccccc} (A \otimes B) \otimes C & \longrightarrow & A \otimes (B \otimes C) & \longrightarrow & (B \otimes C) \otimes A \\ \downarrow & & & & \downarrow \\ (B \otimes A) \otimes C & \longrightarrow & B \otimes (A \otimes C) & \longrightarrow & B \otimes (C \otimes A) \end{array}.$$

(2) For a braided monoidal category, when $\gamma^2 = \mathrm{id}$, the structure is called a *symmetric monoidal category.*
(3) For a symmetric monoidal category, when the associativity and unitality hold strictly, i.e. when $\alpha$, $\gamma$ and $\rho$ gives identities, it is called a *permutative category.*

**Example 2.3.**

(1) $(\mathrm{Set}, \times, \{*\})$, the category of sets with the cartesian product.
(2) $(\mathrm{Set}, \sqcup, \varnothing)$, the category of sets with the disjoint union.
(3) $(\mathrm{Spc}_*, \wedge, S^0)$, the category of pointed spaces with the smash product.
(4) $(\mathrm{Spc}_*, \times, \{*\})$, the category of pointed spaces with the cartesian product.

**Remark 2.4.** Note that not many monoidal categories in nature have strict associativity, i.e. the associator is identity. For example, for $(\mathrm{Set}, \times, \{*\})$, we have

$$(A \times B) \times C \cong A \times (B \times C).$$

But this is not a strict equality, since elements in the left are of the form $((a,b),c)$ and elements in the right are of the form $(a,(b,c))$.

2.1. **Monoidal functors.** Monoidal functors are functors between monoidal categories that preserves the monoidal structures. There are different variants.

(1) *lax monoidal functor* or *monoidal functor*. Such a functor is equipped with a natural transformation

$$\phi_{A,B} : F(A) \otimes F(B) \to F(A \otimes B)$$

satisfying associativity and unitality coherence conditions.

$(a)$ (associativity)

$$\begin{array}{ccccc} (F(A) \otimes F(B)) \otimes F(C) & \longrightarrow & F(A) \otimes (F(B) \otimes F(C)) & \longrightarrow & F(A) \otimes F(B \otimes C) \\ \downarrow & & & & \downarrow \\ F(A \otimes B) \otimes F(C) & \longrightarrow & F((A \otimes B) \otimes C)) & \longrightarrow & F(A \otimes (B \otimes C)) \end{array}$$

$(b)$ (unitality)

$$\begin{array}{ccc} I \otimes F(A) & \longrightarrow & F(I) \otimes F(A) \\ \downarrow & & \downarrow \\ F(A) & \longleftarrow & F(I \otimes A) \end{array} , \quad \begin{array}{ccc} F(A) \otimes I & \longrightarrow & F(A) \otimes F(I) \\ \downarrow & & \downarrow \\ F(A) & \longleftarrow & F(A \otimes I) \end{array} .$$

(2) A op-lax monoidal functor is one where the natural transformation maps in the other direction.
(3) A strong monoidal functor is a lax monoidal functor where the natural transformation is an isomorphism.
(4) A strict monoidal functor is a lax monoidal functor where the natural transformation gives identities.

## 3. Strictification

As mentioned before, monoidal categories are ubiquitous, but not are strict monoidal ones. However,we can always replace a monoidal category with a strict one via strictification.

**Theorem 3.1** ([Isa]). *For any monoidal category* $\mathbf{C}$*, there is a naturally equivalent strict monoidal category* $\mathbf{D}$*.*

We take the free topological monoid to construct a strict monoidal category.

*Proof.* We construct $\mathbf{D}$ as follows.
(1) First, we define the objects to be the free topological monoid generated by the objects of $\mathbf{C}$, and identify the unit of the topological monoid with the unit of $\mathbf{C}$. We write the objects by juxtaposition.
(2) We can define a map $\pi : \mathrm{obj}(\mathbf{C}) \to \mathrm{obj}(\mathbf{D})$:

$$\pi(A_1 A_2 ... A_n) \mapsto (A_1 \otimes (A_2 \otimes (... \otimes (A_{n-1} \otimes A_n)))).$$

(3) The morphisms are defined as follows:

$$\mathrm{Hom}_{\mathbf{D}}(x,y) = \{x\} \times \mathrm{Hom}_{\mathbf{C}}(\pi(x), \pi(y)) \times \{y\}.$$

(4) We extend $\pi$ to a functor by

$$\pi((x,f,y)) = f : \pi(x) \to \pi(y).$$

(5) We can define a functor $\iota : \mathbf{C} \to \mathbf{D}$:

$$\iota(A \xrightarrow{f} B) = (A,f,B).$$

It is straight forward to check that the functors $\pi$ and $\iota$ are monoidal functors, and that $\pi\iota = \mathrm{id}$ and $\iota\pi$ is natural isomorphic to $\mathrm{id}$.

■

Using a similar construction, we can prove that every symmetric monoidal category can be strictified to a permutative category.

## 4. Permutative category

**Definition 4.1.** The *categorical Barratt–Eccles operad* $\tilde{\Sigma}$ is a operad in categories, whose each level is of the form

$$\tilde{\Sigma}(j) = \mathcal{E}\Sigma_j.$$

Here $\mathcal{E}$ : Grp → Cat is the translation functor. The operad action is given by block permutation.

**Proposition 4.2.** *Permutative categories are algebras over the categorical Barratt–Eccles operad.*

*Proof.* The commutativity constraint $\gamma$ is given by the morphisms between two objects in $E\Sigma_2$. ■

**Corollary 4.3.** *The classifying space of a permutative category is a $E_\infty$ space.*

*Proof.* We can take the classifying space levelwise to $\tilde{\Sigma}$ to get the topological operad $B\tilde{\Sigma}$. The classifying space of a permutative category is an algebra over this operad. The operad is $E_\infty$, since $B\mathcal{E}\Sigma_j$ is $\Sigma_j$-free and contractible (it is a model for the universal space $E\Sigma_j$). ■

## 5. Bimonoidal category

The "bi" in the name "bimonoidal category" means that such a category is equipped with two different monoidal structures: an addition and a multiplication.

Sometimes it is called a "rig" category, where "rig" stands for "ring without negatives".

**Definition 5.1** (Symmetric bimonoidal category, [LaP]). A *symmetric bimonoidal category* $\mathbf{C}$ is a category with two symmetric monoidal structures: $(\mathbf{C},\oplus,0)$ for the addition and $(\mathbf{C},\otimes,1)$ for the multiplication, together with natural left and right distributivity natural monomorphisms with components:

$$\delta_l : A \otimes (B \oplus C) \to (A \otimes B) \oplus (A \otimes C),$$
$$\delta_l : (A \oplus B) \otimes C \to (A \otimes C) \oplus (B \otimes C),$$

and natural annihilation isomorphisms with components:

$$a_l : A \otimes 0 \to 0,$$
$$a_r : 0 \otimes A \to 0,$$

and satisfying certain coherence conditions.

We refer the readers to [LaP] for a detailed discussion of the coherence conditions and their relations.

[LaP] only requires the distributivity to be monomorphisms. We focus on when the left distributivity is a natural isomorphism and call such a structure a "strong symmetric bimonoidal category".

**Example 5.2.**

(1) Example 2.3(1) for addition and Example 2.3(2) for multiplication.
(2) There is a type of symmetric bimonoidal categories, where the addition is given by the category-theoretical coproduct, and the multiplication is given by the category-theoretical product. We call such a category a *distributive* category.
- The category of finite sets.
- The category of topological spaces.

**Definition 5.3** (Bipermutative categories)**.** A bipermutative category is a symmetric bimonoidal category where
(1) both monoidal structures are permutative, and
(2) the left distributivity is an natural isomorphism, and the right distrubitivity is identity, and
(3) the following diagram is commutative.

$$\begin{array}{ccc}
(A\oplus B)\otimes(C\oplus D) & \longrightarrow & (A\oplus B)\otimes C\oplus(A\oplus B)\otimes D \\
\| & & \| \\
\| & & (A\otimes C)\oplus(B\otimes C)\oplus(A\otimes D)\oplus(B\otimes D) \\
\| & & \downarrow \\
A\otimes(C\oplus D)\oplus B\otimes(C\oplus D) & \longrightarrow & (A\otimes C)\oplus(A\otimes D)\oplus(B\otimes C)\oplus(B\otimes D)
\end{array}.$$

**Remark 5.4.** One of the coherence conditions one expect a symmetric bimonoidal category to satisfy is the following commutative diagram:

$$\begin{array}{ccc}
A\otimes(B\oplus C) & \xrightarrow{\delta_l} & (A\otimes B)\oplus(A\otimes C) \\
\downarrow{\scriptstyle c_\otimes} & & \downarrow{\scriptstyle c_\otimes\oplus c_\otimes} \\
(B\oplus C)\otimes A & \xrightarrow{\delta_r} & (B\otimes A)\oplus(C\otimes A).
\end{array}$$

By this diagram, the left and right distributivity determines each other. In the definition of bipermutative categories, it is required that left distributivity is an isomorphism and the right one is an identity. It is unreasonable to require both to be isomorphism by the above diagram, since the vertical arrows are only isomorphisms. However, it makes no difference to choose the left to be identity.

## 6. Bistrictification

**Theorem 6.1.** *[Strictification theorem, [May]] Any strong symmetric bimonoidal category is naturally equivalent to a bipermutative category.*

*Proof.* To construct a bipermutative category $\mathbf{D}$ that is equivalent to a strong symmetric bimonoidal category $\mathbf{C}$, we again use the idea of "constructing the free thing and showing the equivalence".

(1) The objects of $\mathbf{D}$ is constructed two steps.
$(a)$ First we take the free topological monoid $M$ generated from obj($\mathbf{C}$) with product $\otimes$, subject to relations $e = 1$ and annihilations. We denote the elements of $M$ by $A_1 \cdot A_2 \cdot \ldots \cdot A_n$ where $A_i$ is in obj($\mathbf{C}$).
$(b)$ Then we take the free topological monoid generated by $M$ with product $\oplus$ subject to the relation $e = 0$. We denote these elements by $x_1 + \ldots + x_n$ where $x_n$ is in $M$.
(2) We can extend $\cdot$ to all objects of $\mathbf{D}$ by

$$(x_1 + \ldots + x_n)\cdot(y_1 + \ldots + y_m) = (x_1 \cdot y_1) + (x_1 \cdot y_2) + \ldots + (x_n \cdot y_m).$$

Here $x_i$ and $y_i$ are in $M$, and $x_i \cdot y_j$ is the monoidal tensor product of $x_i$ and $y_j$ in $M$. The bimonoidal structure is given by $\cdot$ and $+$.

(3) We define a map $\pi_1$ from $M$ to obj($\mathbf{C}$) as in the previous proof, and define a map $\pi : \text{obj}(\mathbf{D}) \to \text{obj}(\mathbf{C})$ by

$$\pi(A_1 + A_2 + ... + A_n) = \pi_1(A_1) \oplus ... \oplus \pi_1(A_n),$$

where $A_i$ is in $M$.
(4) The morphisms are defined to be

$$\mathrm{Hom}_{\mathbf{D}}(x, y) = \{x\} \times \mathrm{Hom}_{\mathbf{C}}(\pi(x), \pi(y)) \times \{y\}.$$

One can check that $\mathbf{D}$ is bipermutative.
(5) The map $\pi$ extends to a functor.
(6) We can define a functor $\iota : \mathbf{C} \to \mathbf{D}$ similarly as in the previous proof.
(7) One can check that both functors preserves the symmetric bimonoidal structures, and $\pi\iota = \mathrm{id}$ and there is a natural isomorphism $\iota\pi \implies \mathrm{id}$. ∎

**Remark 6.2.** In the proof of Theorem 6.1, the constructed category $\mathbf{D}$ only has left distributivity isomorphism but not identity. For example, let $A, B, C, D$ be elements in the original strong symmetric bimonoidal category. Consider the elements $A, B$, and $C + D$ in $\mathbf{D}$. The left distributivity gives

$$(A + B) \cdot (C + D) \to (A + B) \cdot C + (A + B) \cdot D.$$

By definition of $\cdot$, we have

$$\text{LHS} = A \cdot C + A \cdot D + B \cdot C + B \cdot D \neq A \cdot C + B \cdot C + A \cdot D + B \cdot D = \text{RHS}.$$

We have remarked in Remark 5.4 about the asymmetry in distributivity and that either choice works. If one wants left distributivity identity instead of the right, the proof of Theorem 6.1 also needs to be adjusted.

## References

[Isa] John Isabell. *On coherent algebras and strict algebras.* 1969.
[LaP] Miguel Laplaza. *Coherence For Distributivity.* 1971.
[May] Peter May. *$E_\infty$ Ring Spaces and $E_\infty$ Ring Spectra.* 1977.