Talk 11: From symmetric bimonoidal categories to E_∞ ring spectra.

Main reference: J P May The construction of E_∞ ring spaces from bipermutative categories.

As indicated by the title, the goal of this talk is to introduce a method that constructs E_∞ ring spectra from symmetric bimonoidal categories. The following is a route map for this construction.

$$
\begin{tikzcd}[row sep=large]
\text{symmetric monoidal categories} \arrow[d, "\text{strictification}"] & \text{strong symmetric bimonoidal categories} \arrow[d, "\text{strictification}"'] & \text{(Talk 10)} \\
\text{permutative categories} \arrow[d] & \text{bipermutative categories} \arrow[d] & \\
\mathscr{F}\text{-categories} \arrow[d, "B"] & (\mathscr{F}\int\mathscr{F})\text{-categories} \arrow[d, "B"] & \\
\mathscr{F}\text{-space} \arrow[d, shift left, "①"] & (\mathscr{F}\int\mathscr{F})\text{-spaces} \arrow[d, shift left, "③"] & \\
\hat{\mathscr{C}}\text{-spaces} \arrow[u, shift left] \arrow[ddd, shift left, "②"] & (\hat{\mathscr{G}}\int\hat{\mathscr{C}})\text{-spaces} \arrow[u, shift left] \arrow[d, shift left, "④"] & \\
 & (\hat{\mathscr{C}},\hat{\mathscr{G}})\text{-spaces} \arrow[u, shift left] \arrow[d, shift left, "⑤"] & \\
 & & \\
\mathscr{C}\text{-spaces} = E_\infty \text{ spaces} \arrow[uuu, shift left] \arrow[d, shift left, "\text{additive infinite loop space theory}"] & (\mathscr{C},\mathscr{G})\text{-spaces} = E_\infty \text{ ring spaces} \arrow[uu, shift left] \arrow[d, shift left, "\text{multiplicative infinite loop space theory}"] & \\
\text{spectra} \arrow[u, shift left, "\Omega^\infty"] & E_\infty \text{ ring spectra} \arrow[u, shift left, "\Omega^\infty"] &
\end{tikzcd}
$$

Talk 8.

There are two parallel stories. The left one is called the additive story and the right one is called the multiplicative story. The additive story is simpler though they are largely analogous. If we look at the right column, going from the top all the way down to the bottom gives a construction of E_∞ ring spectra out of symmetric bimonoidal categories. As you can tell from the diagram, the construction is complicated and contains many steps. For the sake of time, we will only mention some of the key ideas and skip a lot of the details. Those parallel arrows will induce equivalence of homotopy categories after passing to suitable subcategories. Those are certainly very interesting results, but we won't mention any of those. Also, we will focus on the maps going down (1 2 3 4 5) without saying much about the maps in the opposite direction.

Def. Let $\mathcal{F}$ denote the category of finite based sets $\underline{n} = \{0, 1, \cdots, n\}$, with 0 as basepoint, and based functions. Let $\Pi \subset \mathcal{F}$ be the subcategory whose morphisms are the based functions $\phi : \underline{m} \to \underline{n}$ such that $|\phi^{-1}(j)| \le 1$ for $1 \le j \le n$, where $|S|$ denotes the cardinality of a finite set S. Let $\Upsilon \subset \Pi$ be the subcategory whose morphisms are the based functions $\phi : \underline{m} \to \underline{n}$ such that $|\phi^{-1}(j)| = 1$ for $1 \le j \le n$.

Def. Let $\mathcal{C}$ be an operad. Define a category $\hat{\mathcal{C}}$ by letting its objects be the sets $\underline{n}$ for $n \ge 0$ and letting its space of morphisms $\underline{m} \to \underline{n}$ be

$$\hat{\mathcal{C}}(\underline{m}, \underline{n}) = \coprod_{\phi \in \mathcal{F}(\underline{m}, \underline{n})} \prod_{1 \le j \le n} \mathcal{C}(|\phi^{-1}(j)|)$$

(When $n = 0$, $\hat{\mathcal{C}}(\underline{m}, \underline{0})$ is a point indexed on the unique map $\underline{m} \to \underline{0}$ in $\mathcal{F}$. Units and composition are induced from the unit $\mathrm{id} \in \mathcal{C}(1)$ and the operad structure maps γ.)

Remark. If $\mathcal{C}(j)$ are all nonempty. $\hat{\mathcal{C}}$ is a category of operators.

Example. Consider operad $\mathcal{N}$ with $\mathcal{N}(j) = *$ for all $j \geq 0$.

$\hat{\mathcal{N}} = \mathcal{F}$

If $\mathcal{C}$ is an operad such that $\mathcal{C}(j)$ are all nonempty, the projections $\mathcal{C}(j) \to *$ induces a map $\hat{\mathcal{C}} \to \mathcal{F}$ which is identity on objects and surjective on mapping spaces.

Def A $\hat{\mathcal{C}}$-space Y in $\mathcal{U}$ is a continuous functor $\hat{\mathcal{C}} \to \mathcal{U}$, written $\underline{n} \to Y_n$. Let $\hat{\mathcal{C}}[\mathcal{U}]$ denote the category of $\hat{\mathcal{C}}$-spaces in $\mathcal{U}$.

Now we can define ①: $\mathcal{F}[\mathcal{U}] \to \hat{\mathcal{C}}[\mathcal{U}]$. by composing with $\hat{\mathcal{C}} \to \mathcal{F}$.

Next, we work on ②.

Let $\mathcal{V} = \Upsilon[\mathcal{U}]$ denote the category of Υ-spaces.

For $X \in \mathcal{U}$, define $RX \in \Upsilon[\mathcal{U}]$ such that $(RX)_n = X^n$, the morphisms are the evident projections. For $Y \in \mathcal{T}[\mathcal{U}]$, define $LY = Y_1$. Then we have adjunction.

$$\mathcal{V} = \mathcal{T}[\mathcal{U}] \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{U}$$

Recall that for an operad $\mathcal{C}$, there is a monad C on $\mathcal{U}$ such that the category $C[\mathcal{U}]$ of C-algebras in $\mathcal{U}$ is isomorphic to the category $\mathcal{C}[\mathcal{U}]$ of $\mathcal{C}$ spaces.

Analogously, one can explicitly construct a monad $\hat{C}$ on $\mathcal{V}$ such that the category $\hat{\mathcal{C}}[\mathcal{U}]$ of $\hat{\mathcal{C}}$ spaces is isomorphic to the category $\hat{C}[\mathcal{V}]$ of $\hat{C}$-algebras in $\mathcal{V}$

<u>Prop.</u> $LR = id$, $L\hat{C}R = C$ and the natural map

$$\delta : \hat{C}R \to RL\hat{C}R = RC \text{ is an isomorphism.}$$

By categorical arguments, one can show

<u>Theorem</u> R induces a map $\mathcal{C}[\mathcal{U}] = C[\mathcal{U}] \to \hat{C}[\mathcal{V}] = \hat{\mathcal{C}}[\mathcal{U}]$

One can define map (2) $\Lambda : \hat{\mathcal{C}}[\mathcal{U}] = \hat{C}[\mathcal{V}] \to C[\mathcal{U}] = \mathcal{C}[\mathcal{U}]$ by two-sided monadic bar construction. $\Lambda Y = B(CL, \hat{C}, Y)$.

Next, we move on to the multiplicative story.

Def Let $(\mathscr{C}, \mathscr{G})$ be an operad pair. We can combine the pair of categories $(\hat{\mathscr{C}}, \hat{\mathscr{G}})$ into a single wreath product category $\mathscr{J} = \hat{\mathscr{G}} \int \hat{\mathscr{C}}$. The objects of $\mathscr{J} = \hat{\mathscr{G}} \int \hat{\mathscr{C}}$ are the n-tuples of finite based sets (objects of $\mathscr{F}$) for $n \geq 0$. We write objects as $(n; S)$, where $S = (\underline{s_1}, \cdots, \underline{s_n})$. The space of morphisms $(m; R) \to (n; S)$ in $\hat{\mathscr{G}} \int \hat{\mathscr{C}}$ is

$$\coprod_{\phi \in \mathscr{F}(\underline{m}, \underline{n})} \varepsilon^{-1}(\phi) \times \prod_{1 \leq j \leq n} \hat{\mathscr{C}}\left(\bigwedge_{\phi(i) = j} \underline{r_i}, \underline{s_j} \right), \quad \varepsilon : \hat{\mathscr{G}} \to \mathscr{F}$$

Here we are using smash product of finite based sets, where the empty smash product is $\underline{1}$ and $\underline{m} \wedge \underline{n}$ is identified with $\underline{mn}$ by ordering (a,b) lexicographically.

Note there is a natural map $\mathscr{J} = \hat{\mathscr{G}} \int \hat{\mathscr{C}} \to \mathscr{F} \int \mathscr{F}$, which is identity on objects and surjective on morphisms.

Def A $\mathscr{J}$-space in $\mathscr{U}$ is a continuous functor $Z : \mathscr{J} \to \mathscr{U}$, written $(n; S) \mapsto Z(n; S)$. Let $\mathscr{J}[\mathscr{U}]$ denote the category of $\mathscr{J}$-spaces in $\mathscr{U}$

Analogous to ①, we get ③: $(\mathcal{J}\int\mathcal{J})[\mathcal{U}] \to \mathcal{J}[\mathcal{U}]$

by composing with $\mathcal{J} \to \mathcal{J}\int\mathcal{J}$

Def. A $(\mathcal{N}\int\mathcal{N})$-space is a functor $\mathcal{N}\int\mathcal{N} \to \mathcal{U}$. Write $\mathcal{W} = (\mathcal{N}\int\mathcal{N})[\mathcal{U}]$ for the category of $(\mathcal{N}\int\mathcal{N})$-spaces.

We have adjunctions

$$\begin{tikzcd}
\mathcal{W} \arrow[r, shift left, "L''"] \arrow[rr, bend left=50, "L"] & \mathcal{V} \arrow[l, shift left, "R''"] \arrow[r, shift left, "L'"] & \mathcal{U} \arrow[l, shift left, "R'"] \arrow[ll, bend left=50, "R"]
\end{tikzcd}$$

Changing notations from the additive theory, for $X \in \mathcal{U}$, define $RX \in \mathcal{W}$ to be such that it sends $(0;*)$ to a point and sends $(n;S)$ to $X^{s_1+\dots+s_n}$ for $n \geq 1$.

The L', R' here were previously denoted as L, R in the additive story. For a $\mathcal{N}$-space Y, $R''Y$ is the $(\mathcal{N}\int\mathcal{N})$-space that sends $(0;*)$ to a point and sends $(n;S)$ to $Y_{s_1} \times \dots \times Y_{s_n}$ for $n \geq 0$. Note $R = R''R'$.

For an $(\mathcal{N}\int\mathcal{N})$-space Z, let $L''Z$ be the $\mathcal{N}$-space given by the spaces $Z(1;s)$, $s \geq 0$. Let $LZ = L'L''Z = Z(1;1)$

<u>Def</u> Let $\mathcal{J} = \hat{\mathcal{G}} \int \hat{\mathcal{C}}$ for an operad pair $(\mathcal{C}, \mathcal{G})$. A $(\hat{\mathcal{C}}, \hat{\mathcal{G}})$-space is an object $Y \in \mathcal{V}$ together with a $\mathcal{J}$-space structure on $R''Y$. A map $f: Y \to Y'$ of $(\hat{\mathcal{C}}, \hat{\mathcal{G}})$-spaces is a map in $\mathcal{V}$ such that $R''f$ is a map of $\mathcal{J}$-spaces.

Thus, by definition, the functor $R'': \mathcal{V} \to \mathcal{W}$ embeds the category of $(\hat{\mathcal{C}}, \hat{\mathcal{G}})$-spaces as the full subcategory of $\mathcal{J}$-spaces of the form $R''Y$.

<u>Def</u> One can explicitly construct a monad $\bar{J}$ on the category $\mathcal{W}$ such that the category $\mathcal{J}[\mathcal{U}]$ of $\mathcal{J}$-spaces is isomorphic to the category $\bar{J}[\mathcal{W}]$ of $\bar{J}$-algebras in $\mathcal{W}$.

Define $\tilde{J} = L''\bar{J}R''$.

<u>Prop.</u> We have $L''R'' = id$. The natural map

$\delta'': \bar{J}R'' \to R''L''\bar{J}R'' = R''\tilde{J}$ is an isomorphism

Then $\tilde{J}$ inherits a monad structure from $\bar{J}$. The category of $(\hat{\mathscr{C}}, \hat{\mathscr{G}})$ spaces is isomorphic to the category $\check{\mathscr{J}}[\mathscr{V}]$ of $\check{\mathscr{J}}$-algebras.

[May; The construction of E_∞ ring spaces from bipermutative categories Section 14]

Analogous to ②, we can define ④:

$$\Lambda'' : \mathscr{J}\text{-spaces} = \mathscr{J}[\mathscr{W}] \longrightarrow \check{\mathscr{J}}[\mathscr{V}] = (\hat{\mathscr{C}}, \hat{\mathscr{G}})\text{-spaces}$$

$$Z \longmapsto B(\check{\mathscr{J}}L'', \mathscr{J}, Z)$$

Let $(\mathscr{C}, \mathscr{G})$ be an operad pair. Recall that we have monads C, G so that $\mathscr{C}[\mathscr{U}] = C[\mathscr{U}]$, $\mathscr{G}[\mathscr{U}] = G[\mathscr{U}]$. C induces a monad, also denoted C, on the category $G[\mathscr{U}]$ of G-algebras. The category of $(\mathscr{C}, \mathscr{G})$-spaces is isomorphic to the category of C-algebras in $G[\mathscr{U}]$.

We want a similar definition for $(\hat{\mathscr{C}}, \hat{\mathscr{G}})$-spaces

<u>Ref.</u> Let $\bar{C}$ denote the monad on $\mathscr{W}$ whose algebras are the $\mathscr{N} \int \hat{\mathscr{C}}$-spaces and let $\bar{G}$ denote the monad on $\mathscr{W}$ whose algebras are the $\hat{\mathscr{G}} \int \mathscr{N}$-spaces.

Prop $\bar{C}$, $\bar{G}$ induce monad structures on $L''\bar{C}R''$ and $L''\bar{G}R''$. Moreover, $L''\bar{C}R'' = \hat{C}$.

Def. Let $\check{G} := L''\bar{G}R''$. Let $\check{G}[\mathcal{V}]$ denote the category of $\check{G}$-algebras in $\mathcal{V}$.

Thm There is a distributivity map $\check{\rho} : \check{G}\hat{C} \to \hat{C}\check{G}$ making certain diagrams commute (then we say $\check{G}$ acts on $\hat{C}$). So that $\hat{C}\check{G}$ is a monad on $\mathcal{V}$, the category of $\hat{C}\check{G}$ algebras in $\mathcal{V}$ is isomorphic to the category of $\hat{C}$-algebras in $\check{G}[\mathcal{V}]$. Moreover, there is an isomorphism of monads $\hat{C}\check{G} \to \hat{J}$. So the category of $(\hat{\mathscr{C}}, \hat{\mathscr{G}})$ spaces is isomorphic to the category of $\hat{C}\check{G}$-algebras in $\mathcal{V}$.

Prop. The adjunction (L', R') induces an adjunction

$$\check{G}[\mathcal{V}] \underset{R'}{\overset{L'}{\rightleftarrows}} G[\mathcal{U}]$$

Recall that we have $L'\hat{C}R' = C$, where L', R' were denoted L, R in the additive story. Similar arguments show we have a functor

$$\textcircled{5} : (\hat{\mathscr{C}}, \hat{\mathscr{G}})\text{-spaces} = \hat{C}[\check{G}[\mathcal{V}]] \to C[G[\mathcal{U}]] = (\mathscr{C}, \mathscr{G})\text{-spaces}.$$