The Barratt-Priddy-Quillen Theorem & Algebraic K-theory.

Recall from Folin's talk yesterday.

Thm (Barratt-Priddy-Quillen).

$$CS^0 = \coprod_n B\Sigma_n \xrightarrow{\alpha} \Omega^\infty \Sigma^\infty S^0 = QS^0.$$

is a group completion.

Today: This exhibits QS^0 as the algebraic K-theory space of the symmetric bimonoidal category: $(\mathrm{Finset}, \amalg, \times)$.

§1. A crash course on algebraic K-theory.

Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a symmetric monoidal category.

Defn. $K_0(\mathcal{C})$ = group completion of iso classes of objs in $\mathcal{C}$.

group law = $\otimes$, unit = $\mathbb{1}$.

Examples: $K_0(\mathrm{Finset}, \amalg) = (\mathbb{N}, +)_{gp} = \mathbb{Z}$.

$K_0(\mathrm{Finset}, \times) = (\mathbb{N}, \times)_{gp} = 0.$

$K_0(\text{Findim vect sp over } \mathbb{F}, \oplus) = \mathbb{Z}$.

$K_0(\text{real vector bundles of fin rk over } X) =: KO(X)$

$K_0(\text{Cpx} - - - - - - -). =: KU(X)$.

Let R be a commutative ring

$K_0(R) = K_0(\text{fin gen } \underbrace{\text{proj. } R\text{-mod}}_{\text{vector bundle over spec } R.}, \oplus)$.

$\rightsquigarrow K_1(R) = GL(R)^{ab}$

$\cong GL(R)/E(R)$

where $GL(R) := \operatorname{colim}_n GL_n(R)$

$E(R)$ = subgp generated by elementary matrices

Also explicit definition of K_2, but complicated

Quillen defined higher algebraic groups of R as htpy gps

$K_n(R) := \pi_n BGL(R)^+ \quad n \geq 1$.

· This recovers K_1, K_2.

The plus construction:

X = conn CW cpx, $\quad P \trianglelefteq \pi_1(X)$ perfect normal.

The plus construction of X rel P is.

$f: X \to Y$ s.t.

① on π_1 $\quad 1 \to P \to \pi_1(X) \xrightarrow{f_*} \pi_*(Y) \to 1$.

② on homology gps.

$$f_*: H_*(X; M) \to H_*(\cancel{X}; M)$$

for any local coeff system M.

$= \pi_1(Y)$ - module

Usually we ~~write~~ write $Y = X^+$ when P is the commutator subgp. of $\pi_1(X)$.

Defn: $K(R) = K_0(R) \times BGL(R)^+$.

Thm (Quillen) Let $\mathbb{F}_q$ be a finite field.

Then we have an equalizer diagram.

$$BGL(\mathbb{F}_q)^+ \longrightarrow BU \underset{\psi^q}{\overset{id}{\rightrightarrows}} BU$$

q-th Adams operations

Two consequences:

① . $K_n(\mathbb{F}_q) \cong \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}/q^k - 1 & n = 2k-1 > 0. \\ 0 & \text{else} \end{cases}$

② BU is an Ω^∞-space.

id, ψ^q both Ω^∞-maps.

$\Rightarrow$ $BGL(\mathbb{F}_q)^+$ is an E_∞ space / Ω^∞-space.

Can upgrade the equalizer diagram to:

(*) $$K(\mathbb{F}_q) = \mathbb{Z} \times BGL(\mathbb{F}_q)^+ \longrightarrow \mathbb{Z} \times BU \underset{\psi^q}{\overset{id}{\rightrightarrows}} \mathbb{Z} \times BU = \Omega^\infty ku.$$

$\mathbb{Z} \times BU \simeq \Omega^\infty ku$ is an E_∞-ring space.

Both id and ψ^q are E_∞-ring maps.

$\Rightarrow$ $K(\mathbb{F}_q)$ is an E_∞-ring space!

Question: Is $K(R)$ an E_∞ ring space in general?

Answer: Yes. Can see this from a different construction.

Question from the audience:

How is (*) related to Frobenius?

Answer: Let p be a prime w/ $(p,q)=1$.

Then. $$ku_p^\wedge \simeq K(\overline{\mathbb{F}}_q)_p^\wedge.$$
$$\circlearrowleft$$
$$\text{Fr action} = \psi^q.$$

§2. Algebraic K-theory via group completion

For a category $\mathcal{C}$, let $B\mathcal{C} = |N\mathcal{C}|$.

For $\mathcal{C} = */\!/G$, this recovers the classifying space BG.

Fact:
- When $\mathcal{C}$ has an initial obj. $B\mathcal{C}$ is contractible
- $\mathcal{C} \simeq \mathcal{D} \Rightarrow B\mathcal{C} \simeq B\mathcal{D}$
- If $\mathcal{C}$ is a groupoid, $B\mathcal{C} \simeq \underbrace{\coprod}_{\text{isom class of objs of } \mathcal{C}} B\mathrm{Aut}(c)$.
- $\Rightarrow \pi_0(\mathcal{C}) =$ Isom class of $\mathrm{obj}(\mathcal{C})$

  $\pi_1(\mathcal{C}, c) = \mathrm{Aut}_{\mathcal{C}}(c)$.
- When $\mathcal{C}$ is symmetric monoidal, $B\mathcal{C}$ is a htpy assoc. comm H-space.

For $\mathcal{C}$ symm monoidal groupoid, want to define $K(\mathcal{C})$ s.t.
- $\pi_0 K(\mathcal{C}) = K_0(\mathcal{C})$ defined at the beginning.
- For $\mathcal{C} = (\text{Fin gen Proj. } R\text{-mod.})^{\text{Isom}}$. $K(\mathcal{C}) \simeq K(R)$ defined using "+"-construction.

Idea: Group complete $\mathcal{C}$.

Construction: $\mathcal{C}^{-1}\mathcal{C}$ is a cat w/.

- Obj: $(m, n) \in \mathrm{Obj}\, \mathcal{C} \times \mathcal{C}$. "$m - n$"
- morphism: equivalence class of.

$$(m_1, n_1) \xrightarrow{s \otimes -} (s \otimes m_1,\ s \otimes n_1) \xrightarrow{f, g} (m_2, n_2)$$

by the relation.

$$(m_1, n_1) \xrightarrow{t \otimes -} (t \otimes m_1,\ t \otimes n_1) \xrightarrow{(f', g')} (m_2, n_2)$$

$\simeq$ if $\exists\ \alpha : s \xrightarrow{\sim} t$. making the diagram commute

$$\begin{tikzcd}
(s \otimes m_1, s \otimes n_1) \arrow[r, "f, g"] \arrow[d, "\alpha, \alpha"'] & (m_2, n_2) \\
(t \otimes m_1, t \otimes n_1) \arrow[ur, "f', g'"'] &
\end{tikzcd}$$

Facts:

- $\mathcal{C}^{-1}\mathcal{C}$ is symm monoidal

$$(m_1, n_1) \otimes (m_2, n_2) = (m_1 \otimes m_2,\ n_1 \otimes n_2)$$

- $\mathcal{C} \to \mathcal{C}^{-1}\mathcal{C}$ $\quad m \mapsto (m, e)$ monoidal.

Thm. (Quillen) Suppose $\forall c, c' \in \mathcal{C}$.

- $\mathrm{Aut}(c) \to \mathrm{Aut}(c' \otimes c)$ is injective.

Then $B\mathcal{C} \to B\mathcal{C}^{-1}\mathcal{C}$ is a gp completion

Defn: $K(\mathcal{C}) := B\mathcal{C}^{+}\mathcal{C}$.

From the Ω^∞ machinery:

$K(\mathcal{C})$ is an E_∞-ring space if $\mathcal{C}$ is symm (bi-) monoidal cat.

Example: $\mathcal{C} = \{\text{free } R\text{-mod of fin rk.}\}^{\text{Isom}}$.

The skeleton of $\mathcal{C}$:

$$\cdot R^0 \qquad \cdot R^1 \circlearrowleft BGL_1 \qquad \cdot R^2 \circlearrowleft BGL_2 \qquad \cdot R^3 \circlearrowleft BGL_3 \qquad \cdots\cdots$$

Thm $B\mathcal{C} = \coprod_n BGL_n(R) \longrightarrow \mathbb{Z} \times BGL^+(R)$.

$$BGL_n R \longrightarrow \{n\} \times BGL_n R \longrightarrow \{n\} \times BGL(R) \to \{n\} \times BGL(R)^+$$

is a group completion.

$\Rightarrow B\mathcal{C}^{+}\mathcal{C} \simeq \mathbb{Z} \times BGL^+(R)$.

If $\mathcal{C} = (\text{Fin gen Proj. } R\text{-mod.})^{\text{Isom}}$

$$B\mathcal{C}^{+}\mathcal{C} \simeq K_0(R) \times BGL^+(R).$$

§3 The BPQ thm.

For $\mathcal{C} = (\text{Fin set}, \sqcup)^{\text{Isom.}}$.

$$B\mathcal{C} = \coprod_n B\Sigma_n$$

By BPQ: $\coprod_n B\Sigma_n \longrightarrow QS^0$

is a gp completion.

$\Rightarrow$. $K(\text{Fin Set}) = B\mathcal{C}^+\mathcal{C}$.

$= QS^0$.

The catesian product on Finset gives the E_∞-ring structure on $K(\text{Fin Set})$.

Recall $K(R)$ is E_∞-ring.

$\rightsquigarrow$ get a unit

$$\begin{array}{c} QS^0 \longrightarrow K(R). \\ \wr\| \\ K(\text{Finset}). \end{array}$$

This is induced by a functor.

$$\begin{aligned} \text{Fin set} &\longrightarrow \text{Fin gen Proj. } R\text{-mod}. \\ S &\longmapsto \text{free } R \text{ mod gen by } S. \\ \sqcup &\longmapsto \oplus \\ \times &\longmapsto \otimes. \end{aligned}$$

In good cases, we also get a "+" construction for $K(\mathcal{C})$.

Suppose $\exists$ a seq^n of elements

$$s_1, s_2, s_3 \cdots \quad \text{in } \mathcal{C}.$$

s.t.

- $s_k = a_k \otimes s_{k-1}$ for some $a_k \in \mathcal{C}$.
- $\mathrm{Aut}(s_k) \xrightarrow{a_{k+1} \otimes -} \mathrm{Aut}(s_{k+1})$. is injective.
- $\forall s \in \mathcal{C}$. $\exists s'$ s.t. $s \otimes s' = s_k$ for some k.

Define. $\mathrm{Aut}(\mathcal{C}) := \operatorname{colim} \mathrm{Aut}(s_k)$.

Idea: Think of $\mathcal{C} = \mathrm{FinProj}(R)$. $s_k = R^k \cdots\cdots$

Thm. $K(\mathcal{C}) \simeq K_0(\mathcal{C}) \times B\mathrm{Aut}(\mathcal{C})^+$.

Apply this to BPQ:

$QS^0 \simeq K(\mathrm{Fin\,set}) \simeq \mathbb{Z} \times B\Sigma_\infty^+$

$\Sigma_\infty = \operatorname{colim} \Sigma_n$.

Cor: $\pi_1^S(S^0) = \pi_1(QS^0) = \pi_1(B\Sigma_\infty^+) = (\Sigma_\infty)^{ab} \cong \mathbb{Z}/2$

generated by the equiv class of odd permutations.

QS^0 has finer structure, splittings etc.

Ran out of time here.

Questions from the audience:

- where's the strictification: Symm $\rightsquigarrow$ Perm?

A: This does not change htpy type of $B\mathcal{C}$.

- Galois theory & $K(\mathbb{F}_q)$?

A: $L_{K(1)}K(\mathbb{F}_q)$ is a finite Galois extension of $L_{K(1)}S^0$.

$L_{K(1)}K(-)$ satisfies Galois descent.

- How about BA_∞^+?

A: $B\Sigma_\infty^+ \simeq B\mathbb{Z}/2 \times BA_\infty^+$

$$\Rightarrow \pi_*(BA_\infty^+) \cong \pi_*(B\Sigma_\infty^+) \cong \pi_*^s(S^0) \quad * \geq 2.$$