17 Aug.

§I G-spaces & change of groups

§II G-CW complexes & cellular Theory

§III Equivariant E-M spaces & postnikov Towers

In this talk, G always finite

§I G-spaces & change of groups

Def: 1) A G-space X is a space w/ a continuous G-action

i.e $G \times X \longrightarrow X$
$(g, x) \longmapsto g \cdot x$

$\boxed{G_+ \wedge X \longrightarrow X \text{ based}}$

2) A map $f: X \longrightarrow Y$ between two G-spaces is called G-equivariant or G-map if

$f(gx) = g f(x) \rightsquigarrow \mathrm{Hom}^G(X, Y)$

- Given two G-spaces X, Y base point preserving.

$\rightsquigarrow$ ① $X \times Y$ diagonal

② $\mathrm{Hom}(X, Y)$

$= \{$all maps $X \to Y\}$

with conjugate G-action

i.e $(gf)(x) = gf(g^{-1}x)$

$$\mathrm{Hom}(X, Y)^G = \mathrm{Hom}^G(X, Y)$$

Def: Let Top^G denote the cat of G-spaces & G-maps.

$(\mathrm{Top}^G_*, \wedge, S^0)$

Prop: $(\mathrm{Top}^G, \times, *)$ is a closed sym monoidal category.

Examples:

① Let V be a f.d G-representation. $(G \to O(V))$

$\rightsquigarrow$ several G-spaces

- $D(V)$: unit disk space inside V
- $S(V)$: unit sphere inside V

• $S^V = D(V)/S(V)$: a model of one point compactification.

② finite G-sets

$$\coprod_i G/H_i$$

Def: A G-equivariant homotopy between G-maps

$$f, g : X \longrightarrow Y$$

is a G-map

$$H : X \times I \longrightarrow Y \qquad X \wedge I_+ \longrightarrow Y$$

s.t.

$$\begin{cases} H(x,0) = f(x) \\ H(x,1) = g(x) \end{cases}$$

$\leadsto$ $Ho(Top^G)$ $\qquad$ $Ho(Top^G_*)$

• obj: G-spaces

• Mor: $[X, Y]_G$ = {equivalent homotopy classes of maps from X to Y}.

change of Grps

$H \leq G$ subgp

<u>prop:</u>

$$\begin{tikzcd}
\mathrm{Top}^G \arrow[rr, "\mathrm{Res}_H^G = i^*"] & & \mathrm{Top}^H \arrow[ll, bend right=60, "\mathrm{Ind}_H^G"'] \arrow[ll, bend left=60, "\mathrm{CoInd}_H^G"] \\
\end{tikzcd}$$

$\perp$ $\perp$

$$\mathrm{Ind}_H^G \dashv \mathrm{Res}_H^G \dashv \mathrm{CoInd}_H^G$$

More explicitly, we have the following natural isos:

$$\mathrm{Hom}^G(\mathrm{Ind}_H^G X, Y) \xrightarrow{\sim} \mathrm{Hom}^H(X, \mathrm{Res}_H^G Y)$$

$$\mathrm{Hom}^H(\mathrm{Res}_H^G X, Y) \xrightarrow{\sim} \mathrm{Hom}^G(X, \mathrm{CoInd}_H^G Y)$$

- $\mathrm{Res}_H^G : \mathrm{Top}^G \longrightarrow \mathrm{Top}^H$

  $X \longmapsto \mathrm{Res}_H^G(X)$

  underlying same with only H-action

- $\mathrm{Ind}_H^G : \mathrm{Top}^H \longrightarrow \mathrm{Top}^G$

  $X \longmapsto G \times_H X = G \times X / \sim \; \circlearrowleft G$

$\sim : (gh, x) \sim (g, hx)$

- $\mathrm{coInd}_H^G : \mathrm{Top}^H \longrightarrow \mathrm{Top}^G$

$$X \longmapsto \mathrm{Hom}^H(G, X)$$
$$\{H\text{-equivariant maps } G \to X\}$$
$$(g\cdot f)(g_0) = f(g_0 g)$$

idea of proof:

$$\Psi : \mathrm{Hom}^G(\mathrm{Ind}_H^G X, Y) \longrightarrow \mathrm{Hom}^H(X, \mathrm{Res}_H^G Y)$$
$$f : G\times_H X \longrightarrow Y \longmapsto \Psi(f) : X \longrightarrow \mathrm{Res}_H^G Y$$
$$x \longmapsto f(e,x)$$

$$\Phi : \mathrm{Hom}^H(X, \mathrm{Res}_H^G Y) \longrightarrow \mathrm{Hom}^G(\mathrm{Ind}_H^G X, Y)$$

$$f : X \longrightarrow \mathrm{Res}_H^G Y \rightsquigarrow \Phi(f) :$$
$$\begin{tikzcd} G\times_H X \arrow[r] \arrow[d] & Y \\ G\times_H \mathrm{Res}_H^G Y \arrow[ur, "\text{action}"'] \end{tikzcd}$$
$$(g, x) \longmapsto g f(e,x)$$
$$(g, f(e,x))$$

$$\Phi\Psi(f)(g,x) = g f(e,x) = f(g,x)$$

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actually we can replace $H \leq G$ by a general group homomorphism. $f: H \longrightarrow G$

it induces the pull-back functor

$$f^*: \mathrm{Top}^G \longrightarrow \mathrm{Top}^H$$
$$X \longmapsto X \circlearrowleft \quad h \cdot x = f(h) \cdot x$$

<u>Thm</u>:

$$\begin{tikzcd} \mathrm{Top}^G \arrow[r, "f^*"] & \mathrm{Top}^H \arrow[l, bend right, "f_!"', "\perp"] \arrow[l, bend left, "f_*", "\perp"'] \end{tikzcd} \qquad f_! \dashv f^* \dashv f_*$$

$$f_!: \mathrm{Top}^H \longrightarrow \mathrm{Top}^G$$
$$X \longmapsto {}_G\circlearrowright G \times_H X \qquad (gh, x) \sim (g, f(h)x)$$

$$f_*: \mathrm{Top}^H \longrightarrow \mathrm{Top}^G$$
$$X \longmapsto \mathrm{Hom}^H(G, X) \circlearrowleft G$$

So in particular, ① if f is given by $H \leq G$ inclusion, then we recover the previous adjunctions

② if $f: G \longrightarrow \{e\}$

$$f^*: \mathrm{Top}^e \longrightarrow \mathrm{Top}^G$$
$$X \longmapsto X^{\mathrm{triv}} \text{ with trivial } G\text{-action}$$

& $f_!(X) := e \times_G X = X/G$

$$f_*(X) = \mathrm{Hom}^G(e, X) = X^G$$

i.e. we have the following adjunctions:

$$\mathrm{Hom}(X/G, Y) \xrightarrow{\sim} \mathrm{Hom}^G(X, Y^{\mathrm{triv}})$$

$$\mathrm{Hom}^G(X^{\mathrm{triv}}, Y) \xrightarrow{\sim} \mathrm{Hom}(X, Y^G)$$

based version: $\begin{cases} G_+ \wedge_H (-) \\ \mathrm{Hom}^H(G_+, -) \end{cases}$

Combining $X \circlearrowleft H$ triv

$$\mathrm{Hom}^G(\underbrace{G \times_H X}_{G/H \times X}, Y)$$
$$\simeq \mathrm{Hom}^H(X^{\mathrm{triv}}, \mathrm{Res}^G_H Y)$$
$$= \mathrm{Hom}(X^{\mathrm{triv}}, (\mathrm{Res}^G_H Y)^H)$$

§II G-CW complexes & cellular theory.

<u>Def</u>: A G-CW complex X is a union of G-spaces X_n s.t. X_0 is a disjoint union of orbits G/H_i & Inductively X_{n+1} is obtained from X_n by attaching cells of $G/H \times D^{n+1}$ along the attaching G-maps $G/H \times S^n \to X_n$

i.e

$$\begin{tikzcd} \coprod_\alpha G/H_\alpha \times S^n \arrow[r] \arrow[d] & X_n \arrow[d] \\ \coprod_\alpha G/H_\alpha \times D^{n+1} \arrow[r] & X_{n+1} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Rmk

- S^n & D^{n+1} are equipped with trivial G-actions

- The attaching map

$$G/H_\alpha \times S^n \longrightarrow X_n$$

is determined by $S^n \longrightarrow X_n^{H_\alpha}$

$$\mathrm{Hom}^G(G/H_\alpha \times S^n, X) \simeq \mathrm{Hom}^G(S^n, \mathrm{Hom}(G/H_\alpha, X))$$

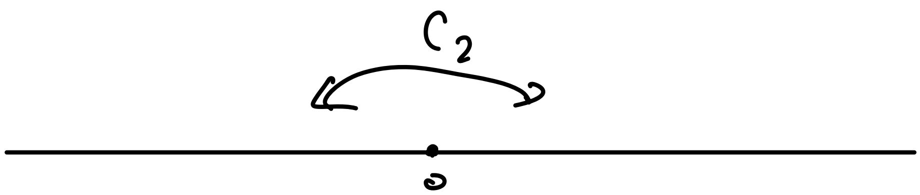
$$= \mathrm{Hom}(S^n, \mathrm{Hom}^G(G/H_\alpha, X))$$

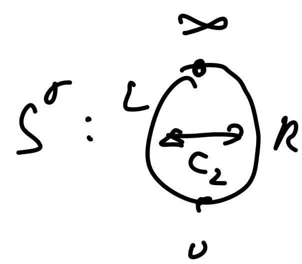
$$= \mathrm{Hom}(S^n, X^{H_\alpha})$$

This indicates the homotopy type of X is actually determined by $[S^n, X_n^{H_\alpha}]$ based

$$\begin{tikzcd} G/H_+ \wedge S^n \arrow[d] \\ G/H_+ \wedge D^{n+1} \end{tikzcd}$$

Example: $G = C_2$, σ: 1-dim sign rep of C_2





Then S^σ has the follow C_2-cw structure

0-cell : $0 \amalg \infty = C_2/C_2 \amalg C_2/C_2$

1-cell : $L \amalg R = C_2/e \times D^1$

attaching map:

$$\begin{tikzcd} C_2 \times S^0 \arrow[r] \arrow[d] & 0 \amalg \infty \arrow[d] \\ C_2 \times D^1 \arrow[r] & S^\sigma \end{tikzcd}$$

Def: For a based G-space & any $H \leq G$, its H-equivariant homotopy group is

$$\begin{aligned} \pi_n^H(X) &:= [G/H_+ \wedge S^n, X]_G \\ &\simeq [S^n, X^H] \\ &= \pi_n(X^H) \end{aligned}$$

Def: A G-map $f: X \longrightarrow Y$ is a weak homotopy equivalence (w.e) if $\forall H \leq G$ subgroup the induced map on fixed points

$$f^H : X^H \longrightarrow Y^H$$

is a w.e

i.e $\pi_*(X^H) \xrightarrow{\sim} \pi_*(Y^H)$

$\forall *$ & any choice of base point.

Thm (Whitehead)

A weak equivalence between G-CW complexes is a homotopy equivalence.

In order to prove this thm, we need some notions & Lemmas.

Recall in non-equivariant case, we say a map $f: Y \to Z$ is an n-equivalence $(n \geq 0)$ if $\pi_*(f)$ is a bijection for $* < n$ & surjection for $* = n$ (for any choice of base point)

Def: Let θ : {conjugacy classes of subgp in G} $\to \{x \in \mathbb{Z} \mid x \geq -1\}$

1) A G-map $f: X \to Y$ is called a θ-equivalence if $\forall H \leq G$ $f^H: X^H \to Y^H$ is a $\theta(H)$-equivalence.
we allow (-1)-equivalence if X^H & Y^H are empty.

2) A G-CW complex is θ-dim if all cells of type G/H has dim $\leq \theta(H)$.

__prop__ (Equivariant HELP)

Let X be a G-CW complex of $\dim \leq \nu$, A is a G-Sub complex of X. And let $e: Y \to Z$ be a ν-equivalence.

Suppose given maps $g: A \to Y$ & $h: A \times I \to Z$,

$f: X \to Z$ s.t the following diagram commutes

$$\begin{tikzcd}
A \arrow[rr, "i_0"] \arrow[dd] & & A \times I \arrow[dl, "h"'] \arrow[dd] & & A \arrow[ll, "i_1"'] \arrow[dl, "g"] \arrow[dd] \\
& Z & & Y \arrow[ll, "e"'] & \\
X \arrow[ur, "f"] \arrow[rr] & & X \times I \arrow[ul, "\tilde{h}"] & & X \arrow[ll] \arrow[ul, "\tilde{g}"']
\end{tikzcd}$$

Then there exists $\underbrace{\tilde{g} \text{ \& } \tilde{h}}_{G\text{-maps}}$ that makes the diagram commutes

idea of proof:

Induction on cells.

$$\begin{tikzcd}
G/H\times S^n \arrow[rr] \arrow[dd] & & G/H\times S^n\times I \arrow[dl] \arrow[dd] & & G/H\times S^n \arrow[ll] \arrow[dl] \arrow[dd] \\
& Z & & Y \arrow[ll] & \\
G/H\times D^{n+1} \arrow[rr] \arrow[ur] & & G/H\times D^{n+1}\times I & & G/H\times D^{n+1} \arrow[ll]
\end{tikzcd}$$

adjunction

$$\begin{tikzcd}
S^n \arrow[rr] \arrow[dd] & & S^n\times I \arrow[dl] \arrow[dd] & & S^n \arrow[ll] \arrow[dl] \arrow[dd] \\
& Z^H & & Y^H \arrow[ll] & \\
D^{n+1} \arrow[rr] \arrow[ur] & & D^{n+1}\times I \arrow[ul, dashed] & & D^{n+1} \arrow[ll] \arrow[ul, dashed]
\end{tikzcd}$$

By assumption, $Y^H \longrightarrow Z^H$ is $\theta(H)$-equivalent.

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Cor: Let $e: Y \to Z$ is a θ-equivalence, then

1) If X has $\dim < \theta$, then the induced map

$$e_*: [X, Y] \longrightarrow [X, Z]$$

is bijective

2) If X has $\dim \theta$

then e_* is surjective.

idea of proof:

1) For surjectivity, we apply $(X, A) = (X, \phi)$

$$\begin{tikzcd}
\phi \arrow[rr] \arrow[dd] & & \phi \times I \arrow[dl] \arrow[dd] & & \phi \arrow[ll] \arrow[dl] \arrow[dd] \\
& Z & & Y \arrow[ll, "e"'] & \\
X \arrow[ur, "f"] \arrow[rr, "i_0"'] & & X \times I \arrow[ul] & & X \arrow[ll, "i_1"] \arrow[ul, "g"']
\end{tikzcd}$$

$g \circ e \simeq f$

2) For injectivity, we apply $(X, A) = (\underbrace{X \times I}, X \times \partial I)$

$\dim(X \times I) \leq Q$

$$\begin{tikzcd}
(f_1 \circ e, f_2 \circ e) \quad X \times \partial I \arrow[rr, "i_0"] \arrow[dd] & & X \times \partial I \times I \arrow[dl, "e"'] \arrow[dd] & & X \times \partial I \arrow[ll, "i_1"'] \arrow[dl] \arrow[dd] \quad (f_1, f_2) \\
& Z & & Y \arrow[ll, "e"'] & \\
X \times I \arrow[ur, "h"] \arrow[rr] & & X \times I \times I & & X \times I \arrow[ll] \arrow[ul, dashed, "\tilde{h}"']
\end{tikzcd}$$

$h : f_1 \circ e \simeq f_2 \circ e$

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proof of whitehead:

$$e : Y \longrightarrow Z \qquad \text{w.e}$$

between CW-complex

$$\rightsquigarrow e_* : [Z, Y] \xrightarrow{\simeq} [Z, Z]$$
$$e_*^{-1}(id_Z) \longleftarrow id_Z$$

We claim: $e_*^{-1}(id_Z)$ is a homotopy inverse of e

By definition

$$e_*^{-1}(id_Z) \circ e \simeq id_Z$$

On the other hand,

$$e \circ e_*^{-1}(id_Z) \circ e \simeq e$$

as maps from $Y \longrightarrow Z$

$$e_* : [Y, Y] \xrightarrow{\simeq} [Y, Z]$$
$$e \circ e_*^{-1}(id_Z) \longmapsto e$$
$$id_Y \longmapsto e$$

This implies $e \circ e_*^{-1}(id_Z) \simeq id_Y$

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§ III: Equivariant E-M spaces & postnikov towers

Def: 1) The orbit category of G $\mathcal{O}_G$ is a full subcat of Top^G on the objects G/H.

2) A coefficient system is a functor

$$M : \mathcal{O}_G^{op} \longrightarrow Ab$$

- will be discuss in details in Ningchuan's talk.

Example: For $n \geq 2$

$$\underline{\pi_n}(X) : \mathcal{O}_G^{op} \longrightarrow Ab \text{ is a}$$

coefficient system.

Recall: In non-equivariant case, Given an abelian grp A (allow A is not abelian when $n=1$) there is a Eilenberg-Maclane space $K(A,n)$ char. by

$$\pi_*(K(A,n)) = \begin{cases} A & * = n \\ 0 & \text{other.} \end{cases}$$

we allow M to be group valued when $n=1$

prop: For each coefficient system M

there is a G-space s.t.

$$\pi_i(K(M,n)) = \begin{cases} M & i=n \\ 0 & i\neq n \end{cases}$$

idea of proof:

we need a result which will be discussed in Ningchuan's talk

thm (Elmendorf)

$$\begin{array}{ccc} \mathrm{Top}^G & \xrightarrow{\simeq} & \mathrm{Fun}(\mathcal{O}_G^{op}, \mathrm{Top}) \\ X & \longmapsto & \underline{X}: \mathcal{O}_G^{op} \to \mathrm{Top} \\ & & G/H \rightsquigarrow X^H \end{array}$$

- This is a Quillen equivalence, or equivalence as ∞-categories

Roughly speaking this means

① the homotopy categories are equivalent

② homotopy colimits & limits behave identically

In other words, these two are same in homotopy theoretic viewpoint.

Let $\Psi : \mathrm{Fun}(\mathcal{O}_G^{op}, \mathrm{Top}) \longrightarrow \mathrm{Top}^G$ be an inverse. Then we consider the following composition

$$\begin{array}{ccccc} \mathcal{O}_G^{op} & \xrightarrow{M} & Ab & \xrightarrow{K(-,n)} & \mathrm{Top} \\ G/H & \rightsquigarrow & \underset{(gp,\, n=1)}{M(G/H)} & \longrightarrow & K(M(G/H), n) \end{array}$$

Then $K(M,n) := \Psi(M \circ K(-,n))$

It's clear $K(M,n)(G/H) = K(M(G/H), n)$

#

Recall: we say a connected space X simple if $\pi_1(X)$ is abelian & $\pi_1(X) \curvearrowright \pi_n(X)$ trivially.

Def: we say a $\underline{\text{G-connected}}$ G-space is simple if $\forall H \leq G$ subgp, its fixed point X^H is simple.

$\mapsto \forall H \leq G \quad X^H$ connected.

Thm. A simple based G-space X has a postnikov towers as follows

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
 & X_2 \arrow[d, "p_2"] \arrow[r, "k^4"] & K(\pi_3(X),4) \\
 & & K(\pi_2(X),2) \arrow[ul] \\
 & X_1 \arrow[d, "p_1"] \arrow[r, "k^3"] & K(\pi_2(X),3) \\
 & & K(\pi_1(X),1) \arrow[ul] \\
X \arrow[r, "\alpha_0"'] \arrow[ur, "\alpha_1"] \arrow[uuur, "\alpha_2"] & X_0 \arrow[r, "k^2"'] & K(\pi_1(X),2)
\end{tikzcd}$$

s.t. $\qquad k^{n+2} \in H_G^{n+2}(X_n;\ \pi_{n+1}(X))$

Bredon cohomology

1) X_0 is a point

2) $\alpha_{n-1} = p_n \circ \alpha_n$

3) $(\alpha_n)_* : \pi_*(X) \longrightarrow \pi_*(X_n)$ is an iso for $* \leq n$ & $\pi_*(X_n) = 0$ for $* > n$

4) $p_n : X_n \longrightarrow X_{n-1}$ is a G-map which is also a principal fibration in the following sense

$$K(\pi_n(X), n) \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow K(\pi_n(X), n+1)$$