Aug 17.

In this talk, we assume G is finite

§I G-spectra & Lewis thm

<u>Def</u>: A G-universe $\mathcal{U}$ contains countably many copies of G-rep s.t

1) $\mathcal{U}$ contains the trivial rep.

2) If $V \in \mathcal{U}$ the so is $\infty V = \bigoplus_{n=1}^{\infty} V$

or $\infty V'$ for each $V' \subseteq V$ sub G-rep

<u>Def</u>: A G-universe is called complete if it contains all irreducible G-representations

<u>Def</u>: Let X be a based finite G-CW complex, Y a based G-space, we define

$$\{X, Y\}_G = \operatorname*{colim}_{\substack{V \subseteq \mathcal{U} \\ \text{f.d}}} [\Sigma^V X, \Sigma^V Y]_G$$

Rmk: By adjunction colimits take

$$[\Sigma^V X, \Sigma^V Y]_G \xrightarrow{\wedge S^{W-V}} [\Sigma^W X, \Sigma^W Y]_G$$

$$\{X, Y\}_G = \operatorname{colim} [X, \Omega^V \Sigma^V Y]_G$$

$$= [X, \operatorname*{colim}_{\substack{V \in \mathcal{U} \\ f.d}} \Omega^V \Sigma^V Y]_G$$

Def: we define

$$QX = \operatorname*{colim}_{\substack{V \subseteq \mathcal{U} \\ f.d}} \Omega^V \Sigma^V X$$

Thm (Equivariant Freudenthal)

There is a G-rep V_0 s.t. for $\forall$ G-rep V we have

$$[\Sigma^{V_0} X, \Sigma^{V_0} Y]_G \xrightarrow{\sim} [\Sigma^{V_0 \oplus V} X, \Sigma^{V_0 \oplus V} Y]$$

i.e.

$$\{X, Y\}_G = [\Sigma^{V_0} X, \Sigma^{V_0} Y]$$

Since $\{V_0 \oplus V, V \in \mathcal{U}\} \subseteq \mathcal{U}$ is cofinal.

Def: The G-Spanier-Whitehead category SW^G has

- objs: based finite G-CW complexes
- mor:

$$SW^G(X, Y) = \{X, Y\}_G$$

pros:

- The objects are easy to work with

- we have spanier-whitehead duality in this category

More precisely, if $X \hookrightarrow S^V$

X is V-dual to unreduced suspension of $S^V \setminus X$ $\qquad D(X) \simeq \Sigma^{-V} S(S^V \setminus X)$

cons:

- It's too small, it is neither complete nor co-complete.

- it is a homotopy category

In non-equivariant world, we have the notions of spectra. We can extend those notions to equivariant world.

<u>Def</u>: 1) A G-prespectrum X consists of based G-spaces $X(V)$ for each $V \in \mathcal{U}$ and based G-maps

$\sigma_{V,W}: \Sigma^{W-V} X(V) \longrightarrow X(W)$ for any $V \subseteq W$, & we require $\sigma_{V,V} = id$ and the following diagram commutes

$$\begin{tikzcd}
S^{Z-W} \wedge S^{W-V} \wedge X(V) \arrow[r] \arrow[d] \arrow[dr, phantom, "\cdots"] & S^{Z-W} \wedge X(W) \arrow[d] \\
S^{Z-V} \wedge X(V) \arrow[r] & X(Z)
\end{tikzcd}$$

when $V \subseteq W \subseteq Z$.

2) A morphism $f: X \longrightarrow Y$ of G-prespectra consists of based G-maps

$$f(V): X(V) \longrightarrow Y(V)$$

which commutes with the structure maps σ.

we denote the category of G-prespectra as PSp^G

3) A G-spectrum X is a prespectrum whose

adjoint structure maps

$$\tilde{\sigma} : X(V) \xrightarrow{\cong} \Omega^{W-V} X(W)$$

are homeomorphisms of G spaces

& we denote the cat of G-spectra as Sp^G

prop:

$$\begin{tikzcd} PSp^G \arrow[r, shift left, "L"] \arrow[r, phantom, "\perp"] & Sp^G \arrow[l, shift left, "U"] \end{tikzcd}$$

existence of L comes from fred adjoint functor theorem.

Examples:

① sphere spectrum S^0

$$S^0(V) := S^V$$

structure map

$$\sigma_{V,W} : S^{W-V} \wedge S^V \longrightarrow S^W$$

② suspension spectrum

For each $A \in \mathrm{Top}^G_*$

$$\rightsquigarrow (\Sigma^\infty A)(V) := S^V \wedge A$$

structure map

$$\begin{array}{ccc} S^{W-V} \wedge S^V \wedge A & \longrightarrow & S^W \wedge A \\ \parallel & & \parallel \\ S^{W-V} \wedge (\Sigma^\infty A)(V) & & (\Sigma^\infty A)(W) \end{array}$$

$$\rightsquigarrow \Sigma^\infty : \underline{\mathrm{Top}}^G_* \longrightarrow pSp^G$$
$$A \longmapsto \Sigma^\infty A$$

this functor admits a right adjoint

$$\begin{tikzcd} \Sigma^\infty : \underline{\mathrm{Top}}^G_* \arrow[r, shift left] \arrow[r, phantom, "\perp"] & pSp^G : \Omega^\infty \arrow[l, shift left] \arrow[d, shift right, "L"'] \arrow[d, phantom, "\dashv"] \\ & Sp \arrow[u, shift right, "u"'] \end{tikzcd}$$

$$\Omega^\infty(E) := E(0)$$

We hope to have a nice sym monoidal cat $\mathcal{C}$ of spectra, Moreover, as a point-set model, we hope $\mathcal{C}$ enjoys the following natural properties

1) This category is sym monoidal

2) There is an adjunction

$$\Sigma^\infty : Top_*^G \rightleftarrows \mathcal{C} : \Omega^\infty$$

s.t

① Ω^∞ is a sym monoidal functor

② $\Sigma^\infty S^0$ is the unit in $\mathcal{C}$

③ the unit map $f: X \longrightarrow \Omega^\infty \Sigma^\infty X$

factor through a natural weak equivaleo

$$
\begin{tikzcd}
X \arrow[r, "\eta"] \arrow[dr, "i"'] & \Omega^\infty \Sigma^\infty(X) \arrow[d, dashed, "\simeq\, f"] \\
& QX
\end{tikzcd}
$$

Thm (Lewis)

There is no such a category.

sketch of proof: Take $G=e$. By assumption

$\Sigma^\infty S^0$ is a unit which means $\Sigma^\infty S^0$ is a commutative monoid. A standard result by Moore [May: The Geometry of iterated Loop space prop 3.6] implies when Ω^∞ is monoidal we have

$$\underbrace{\Omega_1^\infty \Sigma^\infty S^0}_{} \simeq \prod_\alpha HA_\alpha$$

path component of the unit in $\Omega^\infty \Sigma^\infty S^0$

image of $S^0 \to \Omega^\infty E$ (non-base pt of S^0)

$\Rightarrow$ the path component of identity map on

QS^0 is $\prod_\alpha HA_\alpha$ ($\Rightarrow \Leftarrow$)

#

- EKMM S-modules : Elmendorf-Kriz-Mandell-May

Rings, modules & algebras in stable homotopy theory

- orthogonal spectra : Mandell-May-Schwede-Shipley

Diagram spaces, diagram spectra, and FSP's

- symmetric spectrum : Hovey-Shipley-Smith

symmetric spectra

None of these models satisfies the ⑬-property!

All these models are equivalent

- Model categories of diagram spectra ; Mandell, May, Schwede, Shipley

- Equivariant orthogonal spectra and S-modules ; Mandell, May

In the following, all constructions are built in prespectra level. & we use PSp^G to denote the category of pre-spectra.

§ II Wirthmüller isomorphism & transfer maps

<u>Def</u>: Let X be a G-prespectrum, for each $H \leq G$ subgp. we define

$$\pi_q^H(X) := \operatorname*{colim}_{\substack{V \in \mathcal{U} \\ \text{f.d}}} \pi_q^H \Omega^V X(V) \qquad q \geqslant 0$$

$$\pi_{-q}^H(X) := \operatorname*{colim}_{\substack{V \supseteq \mathbb{R}^q \\ \text{f.d}}} \pi_0^H \Omega^{V-\mathbb{R}^q} X(V)$$

$\rightsquigarrow$ the homotopy groups form a coefficient system

$$\underline{\pi_*}(X) : \mathcal{O}_G^{op} \longrightarrow Ab$$

Basic idea:
$$\begin{tikzcd} G/H \arrow[r] \arrow[dr, "②"'] & G/K \\ & G/H^g \arrow[u, "①"'] \end{tikzcd} \qquad H^g \leq K$$

① subgp case $H \leq G$ $\qquad i^* : Top^G \to Top^H$

$$res_H^G : \pi_0^G(X) \longrightarrow \pi_0^H(X)$$
$$S^W \to X(W) \rightsquigarrow S^{i^*W} \rightsquigarrow X(i^*W)$$

② Conjugation action $\quad c_g : H \longrightarrow H^g$

$\rightsquigarrow c_g^* : Sp^{H^g} \longrightarrow Sp^H$

$$
\begin{tikzcd}
c_g^* : \pi_0^{H^g}(X) \arrow[rr] \arrow[dr] & & \pi_0^H(X) \\
& \pi_0^H(c_g^* X) \arrow[ur, "l_g"'] &
\end{tikzcd}
$$

where $l_g : c_g^* X \longrightarrow X$ $\quad$ left multiplication.

$\qquad x \longmapsto gx$

Def: A map $f: X \longrightarrow Y$ between G-pre spectra is called stable weak equivalence if $\forall H \leq G$ subgp.

$$f_*^H : \pi_*^H(X) \longrightarrow \pi_*^H(Y)$$

are isomorphisms $\forall * \in \mathbb{Z}$.

Recall:

$$
\begin{tikzcd}
Top_*^G \arrow[rr, "Res_H^G"] \arrow[rr, phantom, bend left=25, "\perp"] \arrow[rr, phantom, bend right=25, "\perp"] & & Top_*^H \arrow[ll, bend right=60, "Ind_H^G"'] \arrow[ll, bend left=60, "CoInd_H^G"]
\end{tikzcd}
$$

In spectra level, we still have those adjunctions.

$$\begin{tikzcd}[column sep=huge]
PSp^G \arrow[r, "Res^G_H"] & PSp^H \arrow[l, bend right=60, "Ind^G_H"', "\perp"] \arrow[l, bend left=60, "coInd^G_H", "\perp"']
\end{tikzcd}$$

$Res^G_H = i^*$

<u>prop</u>: $Ind^G_H \dashv Res^G_H \dashv coInd^G_H$

$$Res^G_H : PSp^G \longrightarrow PSp^H$$
$$X \rightsquigarrow Res^G_H(X)$$

$Res^G_H(X)(i^*V) := i^*X(V)$

V : a G-rep

there is a subtle change of universe issue:

$i^*\mathcal{U}_G$ cannot run through all rep in H-universe.

$$I^{\mathcal{U}_H}_{i^*\mathcal{U}_G} : PSp^H_{i^*\mathcal{U}_G} \xrightarrow{\simeq} PSp^H_{\mathcal{U}_H}$$

This is because any H-rep can embed into a larger H-rep which underlies a G-rep.

$$\mathrm{Ind}_H^G : pSp^H \longrightarrow pSp^G$$

$$\bar{X} \longmapsto G_+ \wedge_H \bar{X}$$

V : a G-rep

where $G_+ \wedge_H \bar{X}(V) := G_+ \wedge_H \bar{X}(i^* V)$

The structure map is given as follows:

B : H-space

A : G-space

$$\begin{array}{ccc}
(G_+ \wedge_H B) \wedge A & & (g, b, a) \\
\downarrow \wr & & \downarrow \\
G_+ \wedge_H (B \wedge i^* A) & & (g, b, g^{-1} a)
\end{array}$$

$$\begin{array}{c}
(G_+ \wedge_H \bar{X}(i^* V)) \wedge S^{W-V} \\
\downarrow \text{ shearing iso} \\
G_+ \wedge_H (\bar{X}(i^* V) \wedge S^{i^* W - i^* V}) \longrightarrow G_+ \wedge_H \bar{X}(i^* W)
\end{array}$$

Similarly:

$$\mathrm{CoInd}_H^G : pSp^H \longrightarrow pSp^G$$

$$\bar{X} \longmapsto \mathrm{Hom}^H(G_+, \bar{X})$$

where

$$\mathrm{Hom}^H(G_+, \bar{X})(V) = \mathrm{Hom}^H(G_+, \bar{X}(i^* V))$$

The structure maps are defined similarly.

Moreover, we have a natural map

$$\tilde{\Psi} : G_+ \wedge_H X \longrightarrow \mathrm{Hom}^H(G_+, X)$$

which is defined levelwise as follows

$$G_+ \wedge_H X(i^* V) \longrightarrow \mathrm{Hom}^H(G_+, X(i^* V))$$
$$(g, x) \longmapsto \tilde{\Psi}(g, x)$$

$$\tilde{\Psi}(g,x)(\gamma) = \begin{cases} \gamma g x & \text{if } \gamma g \in H \\ * & \text{if } \gamma g \notin H \end{cases}$$

<u>Thm</u> (Wirthmüller)

The natural map $\tilde{\Psi}$ is a stable weak equivalence.

In algebra: $\mathrm{Ind}_H^G M \simeq \mathrm{CoInd}_H^G M$

Cor: In underlying case, it recovers

$$\bigvee_{i=1}^{n} X_i \xrightarrow{\simeq} \prod_{i=1}^{n} X_i$$

Construction of transfer maps:

Given subgps $K \leq H \leq G$

$\rightsquigarrow$ G-map $\quad G/K_+ \longrightarrow G/H_+$

which gives $res^H_K : \pi_*^H(X) \longrightarrow \pi_*^K(X)$

In stable world, we have a "wrong way" map

$$tr^G_H : \quad G/H_+ \dashrightarrow G/K_+$$

lives in pSp^G. i.e

$$tr^G_H : \Sigma^\infty_+ G/H \longrightarrow \Sigma^\infty_+ G/K$$

idea of construction: $\qquad W$: H-rep

Consider $j : H/K \longrightarrow W$ H-equivariant

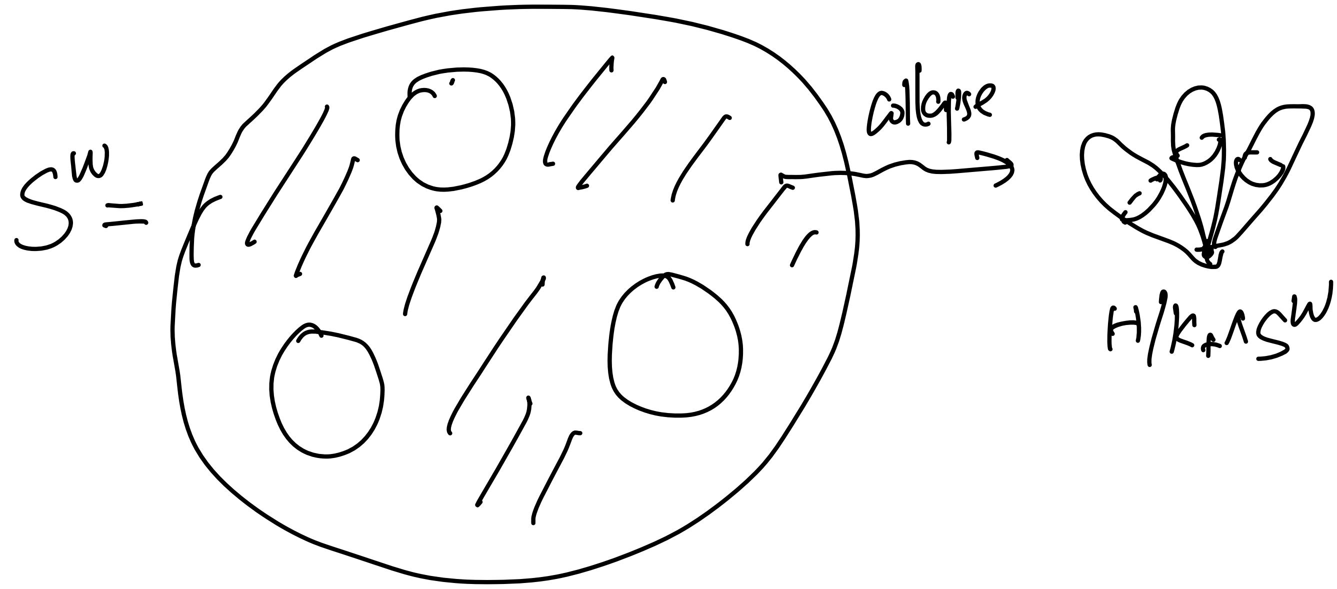
We can assume without losing any generality that the open unit balls around images $j(hK)$ are pairwise

$w = j(eK)$

disjoint.

$$\longrightarrow H/K \times D(W) \longrightarrow W$$
$$[h, x] \longmapsto h(w + h^{-1}x)$$

Then the Thom-Pontryagin construction



gives
$$S^W \longrightarrow H/K_+ \wedge S^W$$

We need $\Sigma^\infty_+ G/H \longrightarrow \Sigma^\infty_+ G/K$

we enlarge W s.t. it underlies a G-rep.

$$\begin{tikzcd}
G_+ \wedge_{H_+} S^W \arrow[r] \arrow[d, "\wr"] & G_+ \wedge_{H_+} (H/K_+ \wedge S^W) \arrow[d, "\wr"] \\
G/H_+ \wedge S^W \arrow[r] \arrow[d, phantom, "\shortparallel"] & G/K_+ \wedge S^W \arrow[d, phantom, "\shortparallel"] \\
\Sigma^\infty_+ G/H (W) & \Sigma^\infty_+ G/K (W)
\end{tikzcd}$$

Rmk: This construction doesn't depend on the choice of embeddings.

We apply the transfer construction to give an inverse of Ψ

Consider $tr_H^G : S^0 \longrightarrow G/H_+ \wedge S^0$

Then

$$\begin{tikzcd}
\mathrm{Hom}^H(G_+, \bar{X}) \arrow[r] \arrow[ddr, "\Phi"'] & G/H_+ \wedge \mathrm{Hom}^H(G_+, \bar{X}) \arrow[d, "\cong"] \\
 & G_+ \wedge_H \mathrm{Res}^G_H \mathrm{Hom}^H(G_+, \bar{X}) \arrow[d, "\mathrm{id} \wedge \varepsilon"] \\
 & G_+ \wedge_H \bar{X}
\end{tikzcd}$$

#

transfer maps on homotopy groups.

$$\mathrm{tr}_H^G : \pi_0^H(X) \longrightarrow \pi_0^G(G_+ \wedge_H X) \longrightarrow \pi_0^G(X)$$

$\rightsquigarrow$ $\underline{\pi}_*(X)$ is a Mackey functor