Equivariant Bredon (co)homology.

§1. The orbit category & Elmendorf thm.

§2. Coefficient systems & Bredon (co)homology.

§3 Axioms and First computations.

§1. The orbit category & Elmendorf thm.

Recall from Zhipeng's talk.

$$G\text{-}Top = Fun(*//G, Top).$$

Today we give a different model that leads to Bredon (co)homology.

Def. Let G be a fin gp.

$$Orb\,G = \begin{cases} \text{obj: } G/H, \ H \leq G \text{ subgp.} \\ \text{mor: } Hom_G(G/H, G/K). \end{cases}$$

More precisely:

$$Hom_G(G/H, G/K) = (G/K)^H ; \quad \phi \text{ unless } g^{-1}Hg \leq K \text{ for some } g.$$

$$\Psi$$

why? f, then $f(eH) = gK$.

s.t. $hgK = gK \quad \forall h \in H$.

$\Rightarrow g^{-1}Hg \leq K$.

[For G a topological gp., should take

$\mathrm{Orb}_G \underset{\mathcal{O}_G}{=} \{$ obj: G/H, $H \leq G$ closed subgp. $\}$]

Defn: An $\mathcal{O}_G$-space is a contravariant functor $\underline{X}: \mathrm{Orb}_G^{op} \longrightarrow \mathrm{Top}$.

Denote the category by $\mathcal{O}_G-\mathrm{Top}$

htpy equivalence = levelwise htpy equivalence.

Observation: Fixed points of a G-space gives a $\mathcal{O}_G$-space.

$$\Phi: G\text{-Top} \longrightarrow \mathcal{O}_G\text{-Top}$$

$$X \longmapsto \underline{X}: (G/H \longmapsto X^H).$$

check: if $g^{-1}Hg \leq K$

then $X^K \subseteq X^{g^{-1}Hg} = g^{-1}X^H$

$$\Theta: \mathcal{O}_G\text{-Top} \longrightarrow G\text{-Top} \quad \xrightarrow{g} X^H.$$

$$\underline{X} \longmapsto \underline{X}(G/e).$$

$\mathcal{O}_G := \mathrm{Hom}_G(G/e, G/e)$.

Observation: $\Theta \dashv \Phi$. Check:

$$\mathrm{Map}_G(\Theta(\underline{X}), Y) \cong \mathrm{Map}_{\mathrm{Orb}_G}(\underline{X}, \Phi(Y)).$$

i.e.

$$\begin{array}{ccc} X(G/e) & \longrightarrow & Y \\ \uparrow & & \uparrow \\ X(G/H) & \overset{\exists}{\dashrightarrow} & Y^H \end{array}$$

since H acts on $X(G/H)$ trivially.

Moreover: $\Theta \Phi = \mathrm{Id}$.

Thm (Elmendorf) $\exists$ a functor

$$\Psi: \mathcal{O}_G\text{-Top} \longrightarrow G\text{-Top}$$

& nat isom $\varepsilon: \Phi\Psi \to \mathrm{id}$ s.t. $\forall\, \underline{X}$

$\varepsilon: \Psi\underline{X}(G/H)^H \longrightarrow \underline{X}(G/H)$ is a htpy equivalence.

If X is a G-CW, then

$$[X, \Psi \underline{Y}]_G \cong [\Phi X, Y]_{\mathrm{orb}_G}$$

$\Phi \dashv \Psi$

Construction: $S: \mathrm{Orb}_G \xrightarrow{\text{underlying space}} \mathrm{Top}$

set $\Psi(\underline{X}) = B(\underline{X}, \mathrm{orb}_G, S)$ ← Two sided bar

$B_n(\underline{X}, \mathrm{orb}_G, S) = \{ x, d_0 \xleftarrow{f_0} d_1 \leftarrow \cdots \leftarrow d_n, s \}$

$x \in \underline{X}(d_0)$

$s \in S(d_n)$

In the end
$$\begin{tikzcd} h\mathcal{O}_G\text{-Top} \arrow[r, "\Phi"] \arrow[r, bend left=50, leftarrow, "\Theta"] \arrow[r, bend right=50, leftarrow, "\Psi"'] & hG\text{-Top}. \end{tikzcd}$$

§2. Coefficient system and Bredon (co)homology.

Nonequivariantly: coeff is Ab.

& cohomology. $h\mathrm{Top}^{op} \to Ab$.

Equivariantly: For cohomology

~~A~~ coeff system is $\underline{M} : \mathrm{Orb}_G^{op} \to Ab$.

Ex: Given any $\mathcal{O}_G$-Top, $\underline{X}$.

functor $\mathrm{Top} \xrightarrow{F} Ab$ (e.g. π_*).

we obtain a coeff system.

$$\mathrm{Orb}_G^{op} \xrightarrow{\underline{X}} \mathrm{Top} \xrightarrow{F} Ab.$$

e.g. For any G-space

$\underline{\pi_*}(X)(G/H) := \pi_*(X^H)$ is a coeff system.

For a G-CW cpx X, define its n-th cellular chain as the coeff system

$$\underline{C_n}(X) := \underline{H_n}(X_n, X_{n-1}; \mathbb{Z}).$$

The connecting homomorphism of the triple (X_n, X_{n-1}, X_{n-2}) gives the differential.

$$\xrightarrow{\delta} \underline{C_n}(X) \xrightarrow{\delta} \underline{C_{n+1}}(X) \xrightarrow{\delta} \cdots$$

Define the Bredon cohomology of X w/ coeff system $\underline{M}$ as.

$$H^*_G(X; \underline{M}) := H^*(\mathrm{Hom}_{\text{coeff}}(\underline{C_\bullet}(X), \underline{M})).$$

Ex: $X = G/H$

$$\underline{C_n}(X)(G/K) = \begin{cases} \mathbb{Z}[G/H]^K = \mathrm{Hom}_G(\mathbb{Z}[G/K], \mathbb{Z}[G/H]) & n=0. \\ 0 & \text{else}. \end{cases}$$

$$H^*_G(G/H, \underline{M}) = \begin{cases} \mathrm{Hom}_{\text{coeff}}(\underline{\mathbb{Z}[G/H]}, M) & n=0 \\ 0 & \text{else}. \end{cases}$$

Claim: $\mathrm{Hom}_{\text{coeff}}(\underline{\mathbb{Z}[G/H]}, M) \cong M(G/H)$.

$$f \longmapsto f(eH)$$

More precisely if $g: G/H \to G/K$

then.

$$\begin{array}{ccccc} eH & \mathbb{Z}[G/H] & \xrightarrow{g} & \mathbb{Z}[G/K] & gH. \\ \downarrow & \downarrow & \wr & \downarrow & \\ & f(eH) & \xrightarrow{g} & g(f(eH)). & \end{array}$$

For homology, need a covariant coeff.

$\underline{N} : \mathrm{Orb}_G \longrightarrow \mathrm{Ab}$.

For $\underline{M} : \mathrm{Orb}_G^{op} \to \mathrm{Ab}$

define $\underline{M} \otimes_{\mathrm{Orb}} \underline{N} = \bigoplus_{H \leq G} M(G/H) \otimes N(G/H) / \simeq$

s.t. $\forall f : G/H \to G/K$.

$n \in N(G/H)$.

$m \in M(G/K)$.

$f^* m \otimes n = m \otimes f_* n$.

This is a coend construction.

$H_*^G(X; \underline{N}) := H_*(\underline{C}_*(X) \otimes_{\mathrm{Orb}} \underline{N})$.

Ex: $X = G/H$.

$\underline{\mathbb{Z}[G/H]} \otimes_G \underline{N} = \bigoplus_{K \leq G} \mathbb{Z}[G/H]^K \otimes N(G/K) / \simeq$.

If $\mathbb{Z}[G/H]^K \neq 0$

then $\mathbb{Z}[G/H]^K$ is gen by the image G/H.

$\underline{\mathbb{Z}[G/H]} \otimes_G N = \mathbb{Z}[G/H] \otimes_G N(G/H) / (g H \otimes n \simeq H \otimes gn)$

$= N(G/H)$.

§3. Axioms and first computations.

Recall the Eilenberg - Steenrod axioms. for nonequivariant (co)homology.

H_*, ∂_*
H^*, δ^*.

- $H_*: hTop \longrightarrow Ab$.
  H^* $hTop^{op} \longrightarrow Ab$.
- Additivity. $H_*(\bigvee X_i) \cong \bigoplus H_*(X_i)$.
  $H^*(\bigvee X_i) \cong \prod H^*(X_i)$.
- LES for cofiber seqn.
- suspension (excision).
- dimension..

Claim: Bredon (co)homology are characterized by the same set of axioms.
except for dimension:

$$\{G/K \mapsto H^G_*(G/K; \underline{N})\} = \begin{cases} \underline{N} & * = 0. \\ 0 & \text{else}. \end{cases}$$

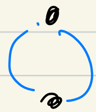
$$\{G/K \mapsto H^*_G(G/K, \underline{M})\} = \begin{cases} \underline{M} & * = 0. \\ 0 & \text{else}. \end{cases}$$

In this sense G/H is a pt in G-equivariant world.

Also, a reduced version.

Example: $G = C_2$.

reflection.

$$X = S^{\sigma} =$$ 

σ: 1-d real sign rep of C_2.

Get a C_2- cofiber seqn.

Nonequivar $S^0 \longrightarrow * \longrightarrow S^1$.

C_2-equivar: $C_2/e \xrightarrow{f} C_2/C_2 \longrightarrow S^{\sigma}$

Let $\underline{M}: \mathrm{Orb}_{C_2}^{op} \longrightarrow \mathrm{Ab}$ be a coeff system. for cohomology.

LES:

$$\to H^n_{C_2}(S^{\sigma}; \underline{M}) \to H^n_{C_2}(C_2/C_2; \underline{M}) \to H^n_{C_2}(C_2/e; \underline{M})$$

$$\to H^{n+1}_{C_2}(S^{\sigma}; \underline{M})$$

$\shortparallel$ 0 unless $n=0$.

when $n=0$

$$0 \to H^0_{C_2}(S^{\sigma}; \underline{M}) \to M(C_2/C_2) \xrightarrow{f^*} M(C_2/e) \to H^1_{C_2}(S^{\sigma}; \underline{M}) \to 0$$

$$\Rightarrow \quad H^0_{C_2}(S^{\sigma}; M) = \ker(f^*)$$

$$H^1_{C_2}(S^{\sigma}; \underline{M}) = \mathrm{coker}(f^*)$$

For homology: $\underline{N}: \mathrm{Orb}_{C_2} \longrightarrow \mathrm{Ab}$.

Check: $H^1(S^{\sigma}; \underline{M}) = \ker f_*: N(C_2/e) \to N(C_2/C_2)$.

$H^0(S^{\sigma}; \underline{M}) = \mathrm{coker} f_*$: - - - - - -.