Summer School on Equivariant Homotopy Theory
Shanghai Center for Mathematical Sciences
August 13–17, 2019

Updated August 7, 2019

Scientific Committee:
Lars Hesselholt, Nagoya University & University of Copenhagen
Peter May, University of Chicago

Organizers:
Agnès Beaudry, University of Colorado Boulder
Mark Behrens, University of Notre Dame
Daniel Isaksen, Wayne State University
Hana Jia Kong, University of Chicago
Zhi Lü, Fudan University
Guozhen Wang, Fudan University
Zhouli Xu, Massachusetts Institute of Technology
Yifei Zhu, Southern University of Science and Technology

1 Day 1: Nonequivariant background

We review the basics of homotopy theory, with the Homotopy Hypothesis in mind. We introduce the notion of infinity categories, and make the point that we should use homotopy limits and colimits in infinity categories. Finally we introduce the universal properties of presheaves in infinity categories, and give the slogan that many of the categories we use, such as spectra and symmetric monoids, are (localizations of) infinity presheaves.

1.1 Talk 1: basic homotopy theory review

1. fiber and cofiber sequences: universal properties, homotopy limits and colimits in general. ([16] 8.4, 8.6, [11] 5.2.2.7)

2. CW complexes: *Top* is generated by one point under colimits. ([16] 10.1, [11] 5.1.5.8)

3. homotopy categories: CW approximation, weak equivalence, Puppe sequence. intuitions from ∞-categories. ([16] 10.4, [11] 1.1)

4. Postnikov towers: truncations as right adjoints. ([11] 1.1.1.4)

1.2 Talk 2: cohomology theories and naive spectra

1. axiomatic cohomology: universal properties of the category *Top*. ([16] 18.1, [11] 5.1.5.6)

2. ordinary cohomology as functors to the derived category, review derived category. ([6] 4.1, [12] 1.3.3)

3. Eilenberg-Maclane spaces, Omega-spectra. ([16] 16.5)

4. Brown representability theorem, represented cohomology theories, representability as the existence of adjoints. ([11] 5.5.2.9)

1.3 Talk 3: basics of spectra

1. Need for good homotopy category of spectra, some history. ([2] III.1, III.2)

2. stable ∞-categories and universal properties of stabilization. ([12] 1.1, 1.4)

3. examples of spectra: K-theory, cobordism, etc. ([2] III.11, I.2)

4. duality: Spanier-Whitehead, Atiyah, Grothendieck, etc. ([2] III.5, III.10)

1.4 Talk 4: monoidal structures

1. monoidal categories: monoids in the symmetric monoidal category of categories. ([12] 2.4.2 (a))

2. ∞-category settings, E_∞-objects. ([12] 2.0.0.7, [14] 3.5)

3. monoidal structure on the category of spectra: smash products, monoidal properties of stabilization. ([2] III.9, [12] 7.1)

4. Day convolution and monoidal structures on presheaves. ([13] 21.6)

5. orthogonal spectra: definition, point of view as presheaves. ([21] 3.1.3, [13] 4.4)

1.5 Evening

simplicial sets, Kan complexes, nerves, quasi-categories. ([11] 1.1, [7] Chapter 1)

2 Day 2: Equivariant homotopy theory

We give basics of equivariant homotopy theory. We introduce Elmendorf's theorem to give the idea that naive equivariant spaces are infinity presheaves over BG and genuine equivariant spaces are presheaves over the orbit category, and use the universal properties of presheaves to understand the axioms of equivariant cohomology theories. The last talk introduces the notion of suspension by a representation sphere, as the preliminaries for equivariant stable homotopy.

2.1 Talk 1: equivariant spaces

1. group actions, G-spaces. ([15] I.1)
2. G-CW complexes: G-spaces are generated by G/H's under colimits. ([15] I.3, [11] 5.1.6.11)
3. fixed point and orbit adjunction. ([15] I.1)
4. homotopy orbits and homotopy fixed points: universal properties, as left/right adjoints of forgetful functor. ([15] V.2, [11] 1.2.13)
5. homotopy groups and the Whitehead theorem, weak equivalence of equivariant spaces. ([15] I.3)

2.2 Talk 2: presheaves on orbit category

1. orbit category: equivariant spaces as presheaves in spaces, the equivalence of homotopy categories. ([15] V.3, [11] 5.1.6.11)
2. universal properties of presheaves. ([11] 5.1.5.6)
3. examples: equivariant cohomology with values in a derived category using universal properties of presheaves. ([15] I.4)

2.3 Talk 3: application

1. ordinary equivariant homology: coefficient systems, axiomatic and Bredon cohomology of G-spaces. ([15] I.4, I.6)
2. Smith theory. ([15] IV.1)

2.4 Talk 4: equivariant spheres

1. orthogonal representations of finite groups, universes of G-representations. ([15] IX.1, IX.2)
2. G-spheres and $RO(G)$-graded homotopy groups. ([15] IX.5)
3. the equivariant suspension theorem, stable homotopy groups. ([15] IX.4, X.6)

4. universal properties of equivariant stabilization, naive equivariant spectra. ([20] 2.1)

2.5 Evening

homology of symmetric groups, plus construction, Barratt-Priddy theorem. ([4] I.5)

3 Day 3: Equivariant stable homotopy theory

We give two points of view for equivariant spectra. First we view them as stabilization of equivariant spaces by all the representation spheres, and give the classical construction using universes, and also Robalo's construction using infinity category techniques. Second we view equivariant spectra as stable infinity sheaves over the enriched Burnside category. The latter makes the introduction of Mackey functors very natural. Finally we introduce orthogonal spectra, and those definitions and theorems using orthogonal spectra.

3.1 Talk 1: introduction to equivariant stable homotopy theory

1. G-prespectra and G-spectra in a universe, stable homotopy groups of G-spectra. ([15] XII.2)
2. Wirthmuller isomorphism, transfers. ([21] 3.2)
3. homotopy fixed points, homotopy orbits: using universal properties, Tate spectra. ([11] 1.2.13, [17] 1.1, [15] XXI.1)
4. fixed points, geometric fixed points: defined using a universe. ([21] 3.3)
5. isotropy separation sequence, characterization of equivariant equivalences via geometric fixed points. ([21] 3.3)

3.2 Talk 2: equivariant spectra as presheaves

1. Burnside category, the isomorphism between the equivariant 0-stem and the Burnside ring. ([15] XVII.2)
2. basics of Mackey functors, the monoidal structure on Mackey functors. Mackey functors as presheaves over Burnside category. ([15] IX.4)
3. presheaves over the Burnside category with values in spectra. ([9] 0.1, [12] 1.4.4.9)

3.3 Talk 3: orthogonal spectra

1. definitions: equivariant setting, global equivariant spectra. ([21] 3.1)
2. suspension spectra, geometric fixed points of suspension spectra. ([21] 3.3)
3. smash products. ([21] 3.5)
4. the tom Dieck splitting (original formulation). ([22] 6.12)
5. the Adams isomorphism; the combination of tom Dieck splitting and Adams isomorphism. ([1], [19])
6. norms (for commutative orthogonal G-ring spectra), the HHR norm functor. ([21] 5.1)

3.4 Talk 4: application

1. equivariant Eilenberg-Maclane spectra: $RO(G)$-graded cohomology, existence of Bredon cohomology, Brown representability, the existence of HM. ([15] XIII.4, XIII.3)
2. Conner conjecture. ([15] 9.6, [18])

3.5 Evening

homological algebra of Mackey functors, equivariant cohomology of a point.

4 Day 4: Advanced topics

We give many examples in equivariant stable homotopy theory. The construction of $MU^{((G))}$ will be treated in detail as the preliminaries for [10].

4.1 Talk 1: equivariant K-theory

1. Borel equivariant theories from non-equivariant spectra. ([21] 4.5.21)
2. connective and periodic equivariant K-theory spectra, comparison with equivariant vector bundles, Greenlees's "equivariant connective K-theory". ([21] 6.3, 6.4, [8])
3. real K-theory. ([3])
4. Atiyah-Segal completion theorem. comparison of fixed points and homotopy fixed points. ([15] XIV.5)

4.2 Talk 2: equivariant cobordism

1. different equivariant flavors of Thom spectra (connective versus orientable). ([21] 6.1)
2. equivariant Thom-Pontryagin constructing and when it is an isomorphism. ([21] 6.2)
3. real cobordism. ([10] 5.2)
4. the construction of $MU^{((G))}$. ([10] 5.1)

4.3 Talk 3: introduction of HHR

([10] Section 1)

1. historical introduction and overview.
2. the main steps in HHR.

4.4 Talk 4: the slice spectral sequence

1. the slices and the slice filtration. ([10] Section 4)
2. Dugger's computation of KR-theory. ([5])
3. the slices of $MU^{((G))}$: the slice theorem, the reduction theorem, and the gap theorem. ([10] Section 6)

5 Day 5: HHR on Kervaire invariant one (up to the experts)

This day is devoted to [10], giving the proofs of the main steps for the Kervaire invariant one problem.

5.1 Talk 1: the slice differential theorem and the periodicity theorem

([10] Section 6, Section 9)

5.2 Talk 2: the homotopy fixed points theorem

([10] Section 10)

5.3 Talk 3: the detection theorem preliminaries

Adams-Novikov, formal A-modules, etc. ([2] III.15, [10] 11.2)

5.4 Talk 4: the detection theorem

proof of the detection theorem. ([10] Section 11)

References

[1] J. F. Adams, *Stable homotopy and generalised homology*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1995, Reprint of the 1974 original. MR1324104

[2] J. F. Adams, *Prerequisites (on equivariant stable homotopy) for Carlsson's lecture*, Algebraic topology, Aarhus 1982 (Aarhus, 1982), Lecture Notes in Math., vol. 1051, Springer, Berlin, 1984, pp. 483–532. MR764596

[3] M. F. Atiyah, *K-theory and reality*, Quart. J. Math. Oxford Ser. (2) **17** (1966), 367–386. MR0206940

[4] F. R. Cohen, T. J. Lada, and J. P. May, *The homology of iterated loop spaces*, Lecture Notes in Mathematics, Vol. 533, Springer-Verlag, Berlin-New York, 1976. MR0436146

[5] D. Dugger, *An Atiyah-Hirzebruch spectral sequence for KR-theory*, K-Theory **35** (2005), no. 3-4, 213–256 (2006). MR2240234

[6] S. I. Gelfand and Yu. I. Manin, *Homological algebra*, Springer-Verlag, Berlin, 1999, Translated from the 1989 Russian original by the authors, Reprint of the original English edition from the series Encyclopaedia of Mathematical Sciences [ıt Algebra, V, Encyclopaedia Math. Sci., 38, Springer, Berlin, 1994; MR1309679 (95g:18007)]. MR1698374

[7] P. G. Goerss and J. F. Jardine, *Simplicial homotopy theory*, Modern Birkhäuser Classics, Birkhäuser Verlag, Basel, 2009, Reprint of the 1999 edition [MR1711612]. MR2840650

[8] J. P. C. Greenlees, *Equivariant connective K-theory for compact Lie groups*, J. Pure Appl. Algebra **187** (2004), no. 1-3, 129–152. MR2027899

[9] B. Guillou, J. P. May, *Models of G-spectra as presheaves of spectra*. arXiv:1110.3571

[10] M. A. Hill, M. J. Hopkins, and D. C. Ravenel, *On the nonexistence of elements of Kervaire invariant one*, Ann. of Math. (2) **184** (2016), no. 1, 1–262. MR3505179

[11] J. Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. MR2522659

[12] J. Lurie, *Higher Algebra*, available at `http://www.math.harvard.edu/~lurie/papers/HA.pdf`.

[13] M. A. Mandell, J. P. May, S. Schwede, and B. Shipley, *Model categories of diagram spectra*, Proc. London Math. Soc. (3) **82** (2001), no. 2, 441–512. MR1806878

[14] J. P. May, *The geometry of iterated loop spaces*, Springer-Verlag, Berlin-New York, 1972, Lectures Notes in Mathematics, Vol. 271. MR0420610

[15] J. P. May, *Equivariant homotopy and cohomology theory*, CBMS Regional Conference Series in Mathematics, vol. 91, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1996, With contributions by M. Cole, G. Comezaña, S. Costenoble, A. D. Elmendorf, J. P. C. Greenlees, L. G. Lewis, Jr., R. J. Piacenza, G. Triantafillou, and S. Waner. MR1413302

[16] J. P. May, *A concise course in algebraic topology*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1999. MR1702278

[17] T. Nikolaus and P. Scholze, *On topological cyclic homology*, Acta Math. **221** (2018), no. 2, 203–409. MR3904731

[18] R. Oliver, *A proof of the Conner conjecture*, Ann. of Math. (2) **103** (1976), no. 3, 637–644. MR0415650

[19] H. Reich and M. Varisco, *On the Adams isomorphism for equivariant orthogonal spectra*, Algebr. Geom. Topol. **16** (2016), no. 3, 1493–1566. MR3523048

[20] M. Robalo, *K-theory and the bridge from motives to noncommutative motives*, Adv. Math. **269** (2015), 399–550. MR3281141

[21] S. Schwede, *Global homotopy theory*, New Mathematical Monographs, vol. 34, Cambridge University Press, Cambridge, 2018. MR3838307

[22] S. Schwede, *Lectures on equivariant stable homotopy theory*, available at `http://www.math.uni-bonn.de/people/schwede/equivariant.pdf`.