

**IWOAT SUMMER SCHOOL 2025:
COMPUTATIONS WITH THE ADAMS SPECTRAL SEQUENCE
AND THE LAST KERRAIRE INVARIANT PROBLEM**

ABSTRACT. In this summer school we will study the Adams spectral sequence, new methods in computing Adams differentials and its applications in resolving the Last Kervaire Invariant Problem [6]. We will study the $H\mathbb{F}_2$ -synthetic spectra and the synthetic Adams spectral sequence. We will see how Adams differentials and extensions interplay with each other in the synthetic setting and how this feature can be used to formulate the Generalized Leibniz Rule and the Generalized Mahowald Trick. In the end, we will discuss the adhoc proof of the Last Kervaire Invariant Problem and its implications in the field of algebraic topology.

The summer school is aimed at senior undergraduate students, graduate students and young researchers in algebraic topology. The school will be held from July 1 to July 5, 2025.

Prerequisites. Stable homotopy group, homological algebra, spectral sequence.

Day 1: Adams spectral sequence. [1] [2] [7]

- (1) Overview of the Kervaire invariant problem.
- (2) Cohomology, Steenrod algebra and Eilenberg-MacLane spectrum. Modules over Hopf algebras, tensor product and dual of A -modules ($H^*(X \wedge Y)$, $H^*(DX)$). Adams spectral sequence. Minimal resolutions and $\text{Ext}^{\leq 1}(X)$. Leibniz rule and naturality. Proofs for $d_2h_i = h_0h_{i-1}^2$.
- (3) May spectral sequence. Elements generated by h_i in $\text{Ext}(S^0)$. Some Massey products on the E_2 -page of Adams (example: $\langle h_0, h_1, h_0 \rangle = h_1^2$). Secondary Steenrod operations and the Adams d_2 -differential.
- (4) Problem session: Compute $\text{Ext}(bo)$ using May spectral sequence or minimal resolution.

Day 2: Extensions. [10] [8] [6]

- (1) Massey products and Toda brackets. Convergence. Moss's theorem.
- (2) Long exact sequences of Ext and π_* . Massey products $\langle h_i, x, y \rangle$ and $\text{Ext}(Ch_i)$. Toda brackets $\langle \alpha_i, \beta, \gamma \rangle$ and $\pi_*(C\alpha_i)$. Matric brackets and finite CW spectra.
- (3) Extension problems. Extension spectral sequences. Crossings. Commutative diagrams and extension spectral sequences.
- (4) Problem session: Find interesting Massey products and Toda brackets using the database of Adams spectral sequences and extension spectral sequences [5].

Day 3: $H\mathbb{F}_2$ -synthetic spectra. [9] [4] [6]

- (1) Overview of synthetic spectra. Adams filtration and λ -divisibility. Synthetic Adams spectral sequence. Rigidity. E_∞ -page of X/λ^n .

- (2) Synthetic extension spectral sequences. Extensions along

$$\Sigma^{0,-n} \nu X / \lambda^{m-n} \xrightarrow{\lambda^n} \nu X / \lambda^m \xrightarrow{\rho_{n,m}} \nu X / \lambda^n \xrightarrow{\delta_{n,m}} \Sigma^{1,-n} \nu X / \lambda^{m-n}.$$

Classical Adams differentials and d^δ -extensions. Extensions on E_r -page and crossings.

- (3) Problem session: Define crossings of Adams differentials on E_r -page.

Day 4: Generalized Leibniz Rule and Generalized Mahowald Trick. [6]

- (1) Generalized Leibniz Rule.
- (2) Generalized Mahowald Trick part 1. May's lemma on triangular categories with symmetric monoidal structure.
- (3) Generalized Mahowald Trick part 2.
- (4) Problem session: Find applications of Generalized Leibniz Rule and Generalized Mahowald Trick in the database. Find applications using Lin's computer program.

Day 5: The Last Kervaire Invariant Problem. [3] [6]

- (1) Quadratic construction. Dyer-Lashof operations and Nishida relations. Finite CW spectra and cell diagrams.
- (2) The inductive method of the Kervaire invariant problem.
- (3) The adhoc proof of the Last Kervaire Invariant Problem.

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- [4] Robert Burklund, Jeremy Hahn, and Andrew Senger. On the boundaries of highly connected, almost closed manifolds. *Acta Math.*, 231(2):205–344, 2023.
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